PROGRESSIVE LESSONS

IN

APPLIED SCIENCE.

PART I.—GEOMETRY ON PAPER.

BY

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INTRODUCTION.

The word Geometry is from the Greek language and means literally the measurement of ground or Land-Surveying; but it has come to be extended in meaning to measurements of many other kinds, and to measurement generally; the simple word *metry*, if it were in use would be a more expressive title for the group of sciences and of processes now described by the title Geometry. In the present work it is held to include measurements of all kinds.

In the science of abstract or pure geometry, all the qualities of matter except its size and shape are put out of view, but in practical or real geometry we cannot omit the consideration of any quality bearing upon the subject in hand; thus the pure geometer pays no attention to the expansion of matter by heat, or to its change of shape by pressure; but the surveyor can neither neglect the stretching of his tape line, nor the temperature of his standard measure. The study of practical geometry, then, must lead us to examine all the physical properties of matter, which can influence our measurements or aid us in conducting them.

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PART I.—GEOMETRY ON PAPER.

LESSON I.

TO DRAW STRAIGHT LINES ON PAPER.

To write on paper we use the blacklead pencil, chalk, or the ink-pen. *Blacklead*, also called *plumbago* and *graphite*, is a softish substance found in various countries, so soft as to be easily cut by the knife: when drawn along the paper it is rubbed down and the powder of it remaining on the paper causes a darkish blue mark. This mark may be removed by means of a piece of indiarubber, or by a bit of not quite dry bread.

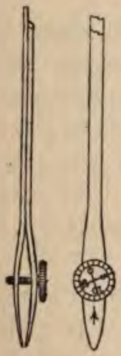
Sometimes the blacklead has in it gritty particles which scratch the paper, and then the marks cannot be quite rubbed out; therefore in choosing a blacklead pencil we should try whether the trace made by it be easily and completely removed by rubber. Good blacklead, for drawing, is scarce and dear, wherefore it is cut into narrow sticks which are fitted into pieces of soft wood glued

together. It is also of various degrees of hardness which are marked by letters as BB, B, H, HH.

Chalk pencils are not much used for geometrical drawings because the lines made by them are not very distinct.

For drawing with the ink-pen we use what is called China or India ink. This consists of a black powder such as lampblack, ground into paste with a little gum, pressed into moulds and dried. For use it is rubbed down in a few drops of water put in a colour-cup or other flat dish; the ink so formed is then filled into the pen by being taken up on a feeder or little stick which is thrust between the plates of the drawing-pen. Other inks and colours may be used, but we must take care that the ink be not such as to corrode (or eat away) the steel of the pen: all the writing inks containing iron or copper do this.

The drawing-pen consists of two narrow pointed plates, usually of steel, brought close together by help of a small screw. The ink is put between them by the feeder until there be nearly as much as will remain when the pen is held upright. If this pen so filled be drawn along the paper so that both points touch, a clean black line is made having its breadth in accordance with the width of the plates. This width can be regulated by means of the screw, so that we are able to draw broad or narrow lines as we may wish.



It is very convenient to have numbered divisions round the edge of the wheel or *head* of the screw, because we can then be sure of making a new line of the same breadth as one that has been made before.

In order to draw a straight line we procure a long piece of wood, bone, ivory, brass or steel having one edge made

quite straight: the paper being laid on a flat table and this rule upon the paper we guide the pencil or the drawing-pen along the edge of the rule.

Although this seem to be an easy operation it must be done with care. The hand must be carried steadily, for if we slope the pen or the pencil differently at different places we shall make an uneven line. In using the drawing-pen we must not press so upon the rule as to close the plates, for then the line will not be of the same breadth all along; nor must we press on the paper so much as to indent it. The outer faces of the steel plates must be kept free from ink lest the edge of the rule be wetted and the line be blurred.

The trace made on the paper is always at a little distance from the rule; we must learn to judge of this distance and must study to keep it the same from end to end of the line. The learner should practise drawing broad and narrow lines with the pen, taking particular care that both points touch the paper, so that the line be clean on both sides.

LESSON II.

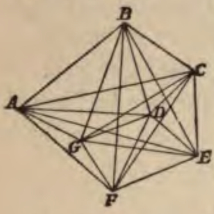
TO JOIN TWO POINTS ON PAPER BY A STRAIGHT LINE.

Two small dots or points having been marked on the paper, we are required to join these by a straight line. For this purpose we bring the straight-edge close up to them, leaving room for the thickness of the pencil or drawing-pen, and then draw the line. It is not easy to do this well at first; we may find that the line has not gone exactly through the points; it may also have stopped short or gone too far. In order to gain expertness in this, the *learner should* sedulously practise the following operation.

LESSON III.

TO JOIN SEVERAL POINTS IN PAIRS BY STRAIGHT LINES.

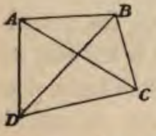
Having made at random a number of small dots upon a sheet of paper we may proceed to join them two and two. Thus if we have made seven points named say A, B, C, D, E, F, G, these may be joined two and two in no



less than twenty-one ways, as A B, A C, A D, A E, A F, A G, B C, B D, B E, B F, B G, C D, C E, C F, C G, D E, D F, D G, E F, E G, and F G; so that, unless three of them happen to be in one straight line, we shall have twenty-one distinct lines.

Having drawn these as carefully as may be, the learner will be sure to observe faults in his first trials; indeed even a practised draughtsman finds it no easy matter to perform well this simple-looking operation.

It may be worth while to note how many lines there are for so many points. For two points there is only one line; for three points there are three lines; for four points, six lines; and so on. The learner would do well to count these and to make a list of the numbers for himself, asking and taking help from no one. If he find out the order or *law* of the successive numbers so much the better.



There is a very well known puzzle; four points as A, B, C, D being marked, to go from one to another of them continuously without going over any line twice.

The student will find an instructive exercise in showing that this is never possible if the number of points be

even, while it is both possible and easy when the number is odd; always provided that no three of the points be in one straight line.

EXERCISE 1.

Take three points not in one straight line, we shall name them A, B, C; join each pair of these by straight lines. In B C mark any point D, join A D; in C A mark some point E, and join B E; the two lines A D, B E cross each other, we shall name the crossing F; join now C F and continue it to meet A B in G; lastly, join G D, D E, E G.

EXERCISE 2.

Draw any two straight lines not meeting each other on the paper; in each of them mark three points, and name these H, I, K in the one, L, M, N in the other, reading in both cases from left to right, or in both cases from right to left. Join H M, I L crossing at P; H N, K L crossing at Q; and I N, K M crossing at R. If the lines be well drawn the three points P, Q, R, will be exactly in one straight line.

LESSON IV.

TO PRODUCE, THAT IS TO LENGTHEN, A STRAIGHT LINE.

WHEN a line already drawn is to be lengthened we have only to bring the straight rule up to it as if we were about to join its two ends, and then to continue it as far as may be desired. This is another instance of an operation seemingly easy but in reality difficult. The eye judges so severely of any inaccuracy in the junction of the two

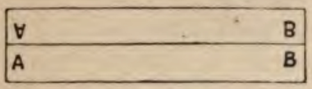
parts or of any alteration in the breadth of the line, that a draughtsman achieves a great success when he can so prolong a line as that the joining is not perceived.

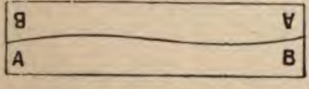
LESSON V.

TO TRY THE STRAIGHTNESS OF AN EDGE.

THE careful geometer's first business is to make sure that his straight-edge is *true*, is really straight, or so nearly straight that he can detect no fault in it.

To test the straightness of the edge A B, we draw, with its help, a line upon the paper; then turning the rule face for face, but not end for end, to the other side of the line we examine whether it again apply; or we draw a second line quite close to the former and examine whether they keep the same distance all along. If any irregularities should appear we mark the projecting places on the rule, and scrape or file them away; repeating the trials until we be satisfied. The mode of making and dressing a straight-edge depends upon the material and will be fully considered in a future lesson. Long continued use wears any straight-edge out of truth and therefore this, or some other, test should be applied from time to time. It will not do to turn the rule end for end because a hollow at the end A might just fit a projection at the other end B; so that rules which are not reversible face for face must be tried against some rule which has *itself been proved* by reversion.

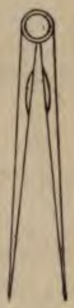




LESSON VI.

TO DIVIDE A STRAIGHT LINE INTO EQUAL PARTS.

FOR ordinary work on paper we use the plain compass or divider, consisting of two legs jointed together, the unconnected ends being nicely pointed. The joint should be stiff enough to resist a moderate strain, but should move steadily without jerks when sufficient pressure is applied; the very points should be rounded, not cornered.



If we have to divide some line A B into, say five equal parts, we judge what would be the fifth part, open the compass to that distance, and beginning at A we step it along towards B five times. If at the fifth time we reach B exactly the division is accomplished; but this is very unlikely; most probably we shall fall short of or go beyond B. In that case we estimate the fifth part of the total error, change the compass by that fifth part and repeat the trial. In this way we come to a division as exact as the circumstances will allow; often, however, after a good many trials.

In the course of these trials the paper is apt to be spoiled by many marks; therefore we prefer to make an exact copy of the distance A B on separate paper whereon our trials may be made.

This kind of compass is convenient in form, but has a very serious defect: its legs are easily bent by slight pressure, so that we may step differently with the same apparent opening, the pressure of the finger on the leg being *enough to* cause a perceptible change in the distance.

For this reason we take care to hold only by the button when repeating a distance: for the same reason we avoid the use of compasses with jointed legs, and particularly those with hair springs, when we desire accurate work. The common spring divider of the workshop is a much more trustworthy instrument; it is composed of two legs connected by a strong spring instead of by a joint, the whole being forged in one piece. When unstrained the legs are far apart; they are held together by a screw and nut. By turning the nut to the right or to the left we close the points or allow them to separate, so that we can manage minute changes much more nicely than with the common compass; while the great stiffness of the legs prevents any perceptible change from side pressure.

When the number of the proposed divisions is large the labour of the trials becomes excessive and we have recourse to various expedients for lessening it. Thus if we have to divide into twelve parts we divide first into three parts, halve each of these and then halve again; and if the proposed number of divisions be fifteen we divide into five, and subdivide each fifth part into three. We can do this for all numbers having divisors, but for indivisible or *prime* numbers such as 17, 19, 23 there is no such help and we must seek for other means.

If it be proposed to divide the line A B into nineteen equal parts we may begin by getting one part as nearly as we can on the trial paper and may mark this distance from *A to C.* The error of the point C is only the nineteenth

part of the total error, and although the total may be quite easily perceived, its nineteenth part may be so small as to be beyond our power of observation; C therefore may be taken as sufficiently exact for our purpose. The remaining portion C B is to consist of eighteen parts, now eighteen is a divisible number, so we divide C B into three equal parts C D, D E, E B; each of these again into three, and each of these smaller parts into two. If we have two dividers we may save trouble by keeping the distance A C in one of them unaltered and using it for the last part of the operation. In such a way schemes may be made to suit other prime numbers.

Exercise 1.

Take any three points, A, B, C, not in one straight line; join A B, B C, C A; halve B C in D; halve C A in E; halve A B in F; join A D, B E, C F. If the work be well done, these three lines meet exactly in one point (say G), and D G is the half of G A, E G the half of G B, and F G the half of G C.

Exercise 2.

Take any three points A B C as before; make B F one third part of B A; C E one third part of C A, and halve B C in D; the three lines A D, B E, C F should all pass through one point G; D G should be one fifth part of D A; E G should be two fifth parts of E B, and F G two fifth parts of F C.

In this case tell what part D G is of G A, and what part E G is of G B.

Exercise 3.

Having made A B C as before, divide B C into seven parts and make C D three of them; make C E one third

part of C A, and B F two fifth parts of B A; then A D, B E, C F should all meet in one point; further, if each of them be divided into nine equal parts, G is a point of division in all.

What part is D G of D A? what part is E G of E B? and what part is F G of F C?

Exercise 4.

A B C being as before, make B D the third part of B C; C E the third part of C A; and A F the third part of A B; join A D, B E, C F; write G at the crossing of A D and C F; H at that of A D and B E; and I at the crossing of B E and C F. Then if each of the lines A D, B E, C F be divided into seven equal parts, G, H, and I are points of division.

What part of D A is D H? What part is H I of B E?

Exercise 5.

Having drawn four lines A B, B C, C D, D A, as in the second figure of Lesson III., without the crossing lines A C, B D; divide A B into three equal parts A E, E F, F B; C D into three equal parts C G, G H, H D; D A into four equal parts D I, I K, K L, L A; and B C into four equal parts B M, M N, N O, O C; join now L M, K N, I O, E H and F G; then each of these five lines is divided into equal parts.

Exercise 6.

The student may vary the above exercise by using any other numbers than 3 and 4.

LESSON VII.

TO MEASURE DISTANCES ON PAPER.

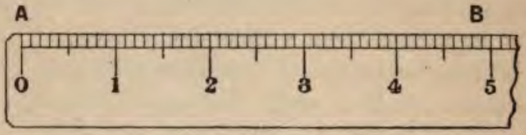
The standard of measure used in Britain is a brass rod carefully preserved in London. Into this two small gold pins are driven, their surfaces made even with that of the rod, and on each of them a small dot is made. The distance between the centres of these dots is declared, by Act of Parliament, to be a yard. For easy reference, copies of the yard are deposited in all the principal cities.

The yard is variously divided for different trade purposes; dealers in cloth divide it into halves, quarters, nails; while artisans in general divide it into 3 feet, and each foot into 12 inches. The foot-rules sold in the shops are almost all very accurately made; most of them are on boxwood, some on ivory, and a good many on steel for the use of smiths and engineers.

For measurements on paper we use scales made thin on the edge so that the divisions may be close to the paper; these are called feather edges or plotting scales. They are usually divided into so many parts per inch and are marked accordingly, thus scale 30 has 30 divisions to the inch; the best of them are divided on a screw dividing engine and are remarkably accurate and cheap. Surveyors need a good number of them to suit the sizes of their plans. They should never be divided alike on both edges because one edge then would be as useful as the two; and the divisions on the two edges of one scale should be so different as not to be easily mistaken. The numbers used by surveyors run from 10 to 80 divisions per inch, and *extend to 12* inches.

For lessons merely a scale 6 inches long of 20, 25 or 30 divisions may suffice.

In order to be easily counted each tenth division is prolonged a good way, and each intermediate fifth division is prolonged half as much, as in the adjoining example. The tens are marked 0, 1, 2, 3, etc., to be read 0, 10, 20, 30, etc. divisions.



The thin graduated edge should never be used as a straight-edge for drawing lines; it would soon be worn and useless. Some 6-inch scales are divided only on one edge, the other edge being left thick for drawing.

To measure the distance between two points as A and B, we bring the divided edge close to them, placing the 0, or *zero* as it is called, opposite to one of them, A, and we are then able to read the number of divisions between it and the second point B. If the second point be not exactly opposite to one of the divisions, we estimate the fraction of a division. It is convenient to imagine the division subdivided into ten parts. The distance A B, as seen in the figure, is thus 48 divisions and by estimation 4 tenth parts of a division, which is written 48·4.

The plan say of a farm is made on a scale of 2 chains to the inch, the length of the usual surveying or Gunter's chain being 66 feet, and being divided into 100 links. In this case ten divisions of the above scale correspond to 1 chain, and the tenth part of a division to 1 link; and the above distance A B would be read 484 links, or 4 chains *84 links.*

If we desire to know the above distance A B actually in inches, we observe that each division of the scale is the twentieth part of an inch, so that the tenth part of a division must be the two-hundredth part of an inch, consequently the distance A B is 484 two-hundredth part of an inch, or as it is written $\frac{484}{200}$, which is just $\frac{242}{100}$ or as written in the way usual for decimal fractions $2 \cdot 42$.

By help of the scale we can also mark off a prescribed distance. Thus if we wish to have a line 46 divisions long, we should first draw in pencil a line more than long enough, and bringing the feather edge to it we should make one dot opposite the zero and another dot opposite the forty-sixth division; the intermediate part may then be drawn in ink, and when the ink is dry the surplus pencil mark may be removed by rubber.

Exercise 1.

Draw a straight line of 4 inches, and divide it into seventeen equal parts.

Draw a second line also of 4 inches, and divide it into fifteen equal parts.

Along a third line make A B equal to one of the former, B C equal to one of the latter divisions, and tell the length of A C in fractions of an inch.

Exercise 2.

Add together the third part of 5 inches and the fifth part of 4 inches.

Exercise 3.

Which is longer, the fifth part of 4 inches or the fourteenth part of 11 inches, and by how much?

LESSON VIII.

ON POLYGONS.

A PORTION of surface is called a *figure;* every figure is bounded by a line or by lines. When the boundary is composed of straight lines the figure is said to be *rectilineal;* the straight lines are called its *sides,* and the points where these lines meet, its *corners.* Figures are named according to the number of the corners, the names being taken from the Greek language; thus a figure having eight corners is called an *octagon,* from *okto* eight and *gonia* a corner. No figure can have fewer than three straight sides with three corners, hence the names are *trigon, tetragon, pentagon, hexagon, heptagon, octagon, enneagon, decagon,* and so on, the general name being *polygon,* from *polus,* many. Instead of the word trigon we very often use the Latin word *triangle,* and for tetragon, *quadrangle,* but it is a pity to spoil the uniformity of the system of names.

A figure, that is a portion of surface, may have more than one boundary; thus the adjoining portion coloured black has three boundaries, an outer and two inner ones. It is a pentagon from which a tetragon and a trigon have been left out; it has twelve corners and twelve sides. At first we shall not pay any attention to such complex figures, and shall consider only simple figures having one boundary and as many corners as sides.

A simple polygon may have all its corners projecting as

in the adjoining hexagon, or it may have re-entrant corners as in the accompanying octagon; and no such polygon can have fewer than three projecting corners.

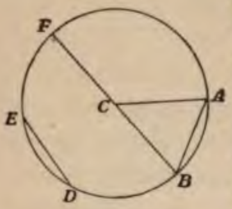
The corners of every trigon (or triangle) project; and every polygon may be divided into trigons by drawing straight lines within the figure from corner to corner, care being taken that these *diagonal* lines, as they are called, never cross each other or the boundary. In the case of simple polygons the number of these trigons is always less by two than the number of corners; but in the case of polygons with more than one boundary this does not hold good, thus the above complex duodecagon would consist of fourteen trigons obtained in this way.

LESSON IX.

TO DESCRIBE A CIRCLE ON PAPER.

The word *circle* is from the Greek *kirkos*. It is a figure every point of whose boundary is at a certain distance from a point called the *centre* (from *kentron* a point or needle). The distance from the centre to the boundary or circumference is called the *radius* (Latin, the spoke of a wheel) and the distance right across through the centre *is called the diameter*, from the Greek *dia* (across) and

metron (a measure). A portion of the circumference of a circle is called an *arc* (*arcus* a bow) and the straight line joining the two ends of the arc is called its chord (Greek *chordi* a gut or gut-string).



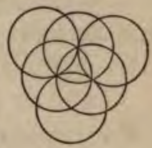
The instrument used for describing (literally *writing down*) circles on paper is a two-legged compass, having one leg nicely pointed and the other arranged to hold a pencil or a drawing-pen. Having set the compass to the required distance we press the centre-point into the paper so as to take hold, and write the boundary by turning the instrument round. The first pressure of the centre-point makes a hole in the paper and the subsequent motion tends to widen that hole, wherefore we must be careful to press lightly lest we make an unseemly mark or even shift the point so that the pen does not return accurately to the beginning of the line. The learner should practise the movement on a piece of waste paper.

If the circle be cut away from the surrounding paper by passing a thin knife-point round the circumference, the circular disc may be turned in its place without changing the position of its centre and a part A B of the circumference would fit accurately to any part D E of equal length: also the disc may be taken out, reversed face for face and again fitted accurately into the old place. This is seen in the axles and bushes of machinery. Also the chord of the arc A B will fit exactly upon that of the equal arc D E, whichever face be uppermost. Hence the diameter D C F divides the circle equally, and each of the arcs B A F and B D E F is a *semi-circumference*.

LESSON X.

TO DIVIDE THE CIRCUMFERENCE OF A CIRCLE INTO EQUAL PARTS.

SINCE equal chords belong to equal arcs, it follows that if we step the dividing compass round the circumference we shall mark off equal portions of it, and hence the division of the circle into equal parts is to be performed just as that of a straight line was done. It will be found convenient to have one or even two copies of the circle made on trial paper. It is very well known and also noteworthy that the radius of a circle is stepped exactly six times round it, so that the division of the circle into six equal parts is very easy. This furnishes us with an excellent exercise for the learner. Let him draw a circle, place the centre point of his compass on the circumference and draw another circle of the same size; place the centre point at one of the crossings, make a third circle, and so proceed covering his paper with circles. If the work be well done all the crossings will be neat without overlap or defect. It is not likely that he will succeed in this at the first or second trial.



If there be two concentric circles, that is, two circles having the same centre; if the outer circumference be divided into equal parts and if radii be drawn to the points of section, these radii divide the inner circumference also equally; for if the larger circle were cut out and turned one division round each radius *would be brought* into the place formerly occupied by

some other radius. When the divisions are to be very numerous and therefore small, so small, say, that the compass cannot be easily managed, we get over the difficulty by making a large circle, dividing it and transferring the divisions to the smaller circle. Thus the teeth of small watch-wheels are cut by fastening the blank discs on the centre of a large divided wheel.

Exercise 1.

Describe a circle with a radius of $1\frac{1}{2}$ inch, and divide its circumference into sixty equal parts, as in a clock face, and mark on it the hours.

Exercise 2.

Divide the circumference of a circle having its radius $1 \cdot 2$ into eight equal parts: divide it also into nine equal parts, beginning at each of the eight divisions. The whole circumference is then divided into seventy-two equal parts.

Exercise 3.

The common year consists of three hundred and sixty-five days; make a circle of $1 \cdot 6$ radius and divide its boundary into three hundred and sixty-five equal parts.

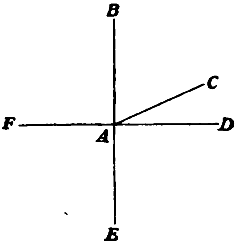
Exercise 4.

Divide the circumference of a circle into three hundred and sixty equal parts.

LESSON XI.

ON ANGLES.

It is of the greatest importance that the student should have a clear idea of what is meant by the word *angle* as used by geometers. The ancient geometer Euclid had not attained to a clear notion on this subject and the imperfection of his idea runs through the whole of his celebrated treatise. He says that an angle is the inclination of two straight lines which meet and which are not in one straight line. This description only shifts the name from *angle* to *inclination* and the limitation leads to a wrong idea. The true idea of *angle* is to be got by considering the turning of a line round upon one of its ends. If the line A B turn, like the hand of a clock, upon the end A as a pivot it will come again into the same position having then made *one turn;* if the motion go on the line will again and again come to the first position having made two, three, four . . . . turns. The *quantity of turning* is what we call *angle.*



In passing from the direction A B into the direction A C the line makes only part of a turn. To come into the position A D it makes one quarter of a turn; to get into the position A E it makes half a turn, to get to A F three quarters of a turn, and to come again to the direction A B it makes a whole turn.

The angle of the two lines A B, A C is the quantity of turning needed to bring a line from the direction A
into the direction AC; it is named by reading a letter

each of its *sides*, putting the letter at the vertex (or point of turning, from *verto* I turn) in the middle as B A C. The angle B A C in the accompanying figure is less than the quarter of a turn; B A D is a quarter turn, B A E is half a turn, B A F three quarters, and we may write B A B for a whole turn. In ordinary geometry we have very little to do with angles of more than one turn, but in mechanics and in astronomy such angles are of very frequent occurrence. Here we see the inconvenience of Euclid's limitation; according to him B A E, which is half a turn, is no angle at all; and he had no idea whatever of such an angle as B A F, three quarters of a turn.

The *angle* has nothing to do with the lengths of the lines, it has only to do with their directions; thus although the lines in the figure were continued to be yards or miles long the angles would be unchanged, just as the hours are shown as well by the hands of a pocket watch as by those of a steeple clock.

LESSON XII.

TO MEASURE ANGLES.

To measure a line is to compare it with a known or standard line which standard is arbitrarily taken. So to measure an angle is to compare it with a known angle; here we have a natural or absolute standard of comparison namely the *whole turn*, which is the same all the world over. The ancient, and even yet the common, practice is to divide the turn into 360 equal parts called degrees, each degree into 60 minutes (minuti primi) and each minute into 60 seconds (minuti secundi); the modern and much

more convenient division, slowly coming into use, is into 400 degrees, each degree being divided decimally. For certain purposes astronomers find it even more convenient to divide the turn into one thousand parts.

For measuring angles on paper we use an instrument called the *protractor*, which is a circular disc of any convenient material, having its edge divided into the proper number of degrees. A piece of cardboard answers, and the student would do well to prepare one for himself; he may even make two, one for the ancient division into 360°, the other for the modern division into 400^c.

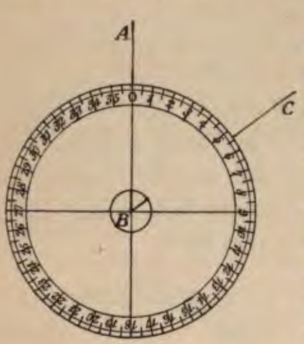
After the divisions have been completed, the circle must be neatly cut along the edge, and the centre hole also made. Two fine threads must then be stretched across, to mark the centre.

Protractors are sometimes made on thin horn, but it is seldom that the horn is quite flat. When made in brass or ivory, the divisions are cut on the turning-lathe or dividing-engine, and the centre mark is sometimes made on a bit of glass fixed in the middle.

It is usual to write the numbers of the degrees in the direction from left to right, that is in the direction usual for the hands of a clock.

In order to measure an angle A B C by help of th

instrument, we bring the centre-mark of the protractor exactly over the vertex B of the angle, and then count how many degrees are included between the two sides.



We may save the trouble of counting by placing the o or beginning of the divisions on one side as AB; the number of degrees may then be read off directly. The beginning of the degrees, and in general the beginning of any scale is called the *zero;* this is from an Arabic word ذرّ or ذرّه zerr, or zerré an-atom. When the Arabs borrowed the present common notation of numbers from India they used a small dot or atom to represent an empty place; the zerré therefore corresponds to the o (an empty ring) used by us for the same purpose.

It does not matter on what part of the circumference the angle may be measured; if the instrument be truly graduated there can be no difference; we have thus a ready means of testing the accuracy of the divisions; by making the measurement on different parts of the protractor and comparing the results. Half circle protractors cannot be thus tested; the complete circle is much to be preferred.

Again, whether we use a small or a large protractor, the number of degrees must be the same; the large protractor, however, has this advantage, that the fractions of a degree may be more easily estimated, seeing that the divisions are l[illegible] If the *protractor* should extend beyond the sides

of the angle, these sides have to be produced sufficiently; a very large protractor is thus inconvenient for work on a small scale.

Exercise 1.

Make an angle of 53°; measure on one side seventy-seven parts from a scale, on the other side seventy-five parts; join the ends, measure the angles there, and measure also the length of the line.

Exercise 2.

Draw A B=57 parts, make the angle A B C = 117°; B C = 53 parts; B C D = 108°; C D = 71 parts. Join A D, A C, B D, and measure them; measure also the angles D A C, C A B, D A B, B C A, A C D, C D B, B D A, C D A, A B D, and D B C.

Exercise 3.

Convert all the measurements of the angles in the above figure (which have been taken in ancient degrees) into centesimal degrees (often called *grades*, from the French, not the English word grades spelt in the same way; the *a* as in gr*a*duate).

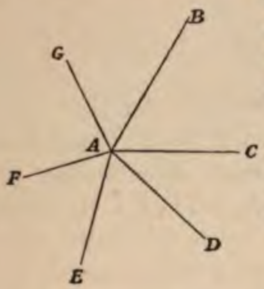
Exercise 4.

Make A B = 55 parts; A B C = 108°; B C = 55 parts; =B C D=108°; C D=55 parts; C D E=108°; and D E=55 parts. Join and measure E A, A D, D B, B E, E C, C A; measure also the angles E A D, D A C, C A B, A B E, and so on. State the angles also in centesimal degrees.

LESSON XIII.

ON THE ADDITION AND SUBTRACTION OF ANGLES.

In turning from the direction A B into the direction A C, a line passes over or *makes* the angle B A C. If the motion be continued until the line come to the direction A D, the additional angle C A D is made; wherefore B A D is the sum of B A C and C A D, or, as it is written concisely



$$BAD = BAC + CAD;$$

wherefore also C A D is the difference between B A D and B A C or

$$CAB = BAD - BAC.$$

In this way the angle B A F is the sum of the four angles B A C, C A D, D A E and E A F. But the same name B A F would do, in the adjoining figure, for the sum of the two angles B A G and G A F made by turning the line from the direction A B into the direction A G and then into the direction A F. Thus the same name B A F is given to two angles one made by turning from right to left, the other by turning from left to right; the first, in the present case, less than a half turn, the other more than a half turn, and both together making a whole turn. Ordinarily there is no inconvenience from this double meaning; but in the practice of surveying great confusion would result, and therefore it becomes necessary to make a distinction. This we do by taking care to name the sides of the angle in the *order of the degrees* on the protractor, thus by G A D we

shall intend the angle made up of the three parts G A B, B A C, C A D; while by D A G we shall mean that made up of D A E, E A F, F A G. If the zero of the protractor be placed on A G the reading at A D will give the former angle, and if the zero be placed on A D, the reading at A G will give the latter.

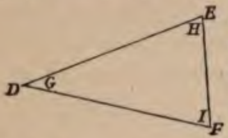
LESSON XIV.

ON TRIGONS OR TRIANGLES.

If three points not all in one straight line be joined two and two, a figure is formed having three sides which, meeting in pairs, form three angles. Such a figure is called a *triangle* from the Latin, a *trigon* from the Greek. The measurement of trigons is called *trigonometry*. A trigon is named by reading the letters placed at its three corners as A B C. But three such letters have been already used to designate an angle, so that there is again confusion; and it becomes necessary, whenever there is any likelihood of mistake, to tell distinctly whether it be an angle or a triangle that is intended. The one is the quantity of turning, the other is the *surface* enclosed by the three straight lines. The accompanying figure has three corners A, B and C; three sides B C, C A and A B; and three angles B A C at A, C B A at B and A C B at C. Each side has an angle opposite to it, and is said to subtend that angle, thus A B is said to *subtend* (outstretch) the angle A C B; the angle A C B *is said to be contained* (that is limited) by the two

sides CA and CB. A trigon having two of its sides of equal length is called *isosceles* (properly *isoskeles* from ισος equal, and σκελος a leg), and when all the three sides are of one length it is equilateral (from the Latin *latus* a side).

An isoskeles trigon may be reversed and replaced. Thus if we draw from the point D two equal lines DE, DF and



join the ends EF, the isoskeles triangle thus formed is reversible face for face. On cutting through the paper along the lines DE, EF, FD and so separating the internal part GHI, we may lift this out, turn it over and replace it, putting I at E, IG along ED, in which case G must be again at D since the lines are of one length; also GH must lie along DF because the angle is unchanged, and consequently H must be at F. Thus we see that the angles FED and DFE must be alike, since the same angle GIH fits to each of them; and we conclude, in general, that whenever two sides of a trigon are alike, the angles opposite to them are alike also.

If, from K as a centre, we sweep a circular arc and in-

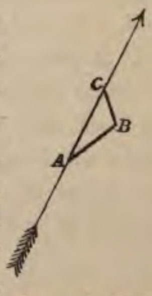
flect in that a distance ML equal to the radius, we easily construct an equilateral trigon KLM; this may be regarded as isoskeles in three ways. Its three angles must then be all alike; and it may be fitted into the same place in six different ways, thrice with one face up and thrice with the other face up.

No angle of a trigon can be so much as half a turn, that is as 180° ancient or 200ᶜ modern degrees; indeed when we measure the three angles and add them together we *find that all* the three just make up a half turn. The

learner should draw a variety of three-sided figures, measure and sum the angles of each; he will find, within the small errors incident to all measurements, that the sum is 180° or 200^c according to the kind of protractor used. This *truth* or *theorem* as it is called, is the most important truth in geometry. For thousands of years writers on speculative geometry have sought to find out *why* this theorem is true, or to show that it *must* be true. In Euclid's, which is the oldest treatise that has come down to us, a series of thirty propositions is occupied with the proof of this theorem, but without success; for at the end recourse is had to an axiom or unproved theorem in which is taken for granted the very thing to be proved. Many trials have since been made, all of them fruitless, so that this truth, the fundamental truth of all geometry, rests solely upon *trial* or *experiment*.

The neatest and clearest illustration of it is that given by the Sicilian philosopher Archimedes, of Syracuse.

Let A B C be the triangle; take a very long thin stick, with the ends marked so as to be recognized and lay it along the side A C noting in which way the marks point. Turn this stick from left to right upon the point A as a vertex until it come to lie along A B, and we shall have turned it through the angle C A B. Still keeping the motion from left to right, turn the stick upon B as a vertex until it lie along B C, and we shall have described the angle A B C; lastly turn it upon C as a vertex through the angle B C A, that is till it be along C A, and we shall find the stick turned half round.

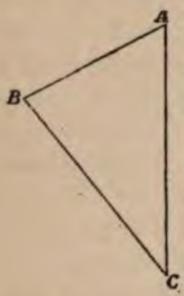


LESSON XV.

TO CONSTRUCT A TRIGON OF WHICH TWO SIDES AND THE ANGLE MADE BY THEM ARE KNOWN.

Let it be proposed to make a triangle having one side 37 divisions, a second side 53 divisions long, while the angle contained by these sides is 78 ancient degrees.

For this we draw a line A B 37 divisions long, measuring the length by help of the scale. By means of the protractor we mark off the angle A B C of 78°, drawing the line in pencil; along this line we next mark off B C 53 divisions, and lastly we join A C.



Here it is to be noticed that the angle of 78° may be laid either to the right of A B as in the figure, or to the left; and that although each of the figures so got have the proposed dimensions, only one of them may be suitable for the purpose in hand; we must therefore consider which face, as it were, is to be upmost; that is to say, whether the angle be A B C or C B A.

It is quite clear that if another triangle be made with the same dimensions it will be in every way like the former, its third side will be equal to A C, and the remaining angles will be equal to C A B and B C A respectively. The one figure would just fit on the other; in this case they are said to be *alike* or *equal* to each other.

Exercise 1.

Make an angle of 52°, measure each side of it eighty-nine parts, join the ends, measure the third side and the two *angles.*

EXERCISE 2.

Make an angle of 119°, with each of its sides sixty-five parts; join the ends; measure the third side and the angles.

EXERCISE 3.

Make A B = 99 parts, A B C = 45° = 50^c, and B C = 70 parts; join A C, measure it and measure the angles A C B, C A B.

EXERCISE 4.

Make D E = 97, D E F = 30°, and E F = 84; join F D, measure it and measure the angles E F D and F D E.

EXERCISE 5.

Construct the triangle G H I having G H = 58, G H I = 83°, H I = 71; measure the third side and the two angles.

EXERCISE 6.

Construct the trigon K L M having K L = 47, K L M = 137°, L M = 83; measure the remaining side and angles.

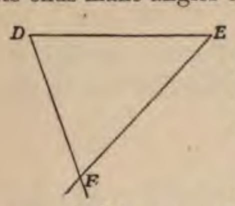
LESSON XVI.

TO CONSTRUCT A TRIGON OF WHICH TWO ANGLES AND A SIDE ARE KNOWN.

WHEN we know the number of degrees in each of two of the angles we may easily discover the third angle because all the three make up half a turn: we have therefore only to add together the two known angles and to deduct the amount from 180° or from 200^c according *to the division used.*

Let it be proposed to make a triangle having one side 57 divisions of the scale, and the angles at its ends 47° and 69° respectively.

For this we draw a line D E 57 divisions long; and at its ends make angles of the given numbers of degrees, continuing the sides of those angles (in pencil) until they cross each other at some point F; D E F is the trigon required.



Any other triangle made with the same dimensions would be exactly a copy of this one; only it may happen to be reversed; hence if two angles and a side of one trigon be equal to the corresponding angles and side of another trigon the figures are alike in every way; the one would fit upon the other.

If, however, the two angles at D and E be equal, the figure could be reversed face for face and the side D F would be equal to E F.

When the proposed angles at D and E make together nearly 180° or 200^c their sides must be continued far in order to meet; if the sum of the angles be more than a half turn the lines widen out; they would need to be continued in the opposite directions in order to meet. And when the proposed angles make just half a turn, their sides cannot meet at all. We shall consider this case in a future lesson.

Exercise 1.

Construct N O P such that O N P = 65°, N P = 53 parts, and N P O = 73°; and measure the two sides N O and P O; find also the angle P O N both by computation and by measurement.

EXERCISE 2.

Make R Q S = 36°, Q S = 55, Q S R = 36°, and measure Q R and R S.

EXERCISE 3.

Make U T V = 45°; T V = 41; T V U = 45°, and measure T U and U V.

EXERCISE 4.

Make Y X Z=97°; X Z=47; X Z Y=63°, and measure X Y, Y Z.

EXERCISE 5.

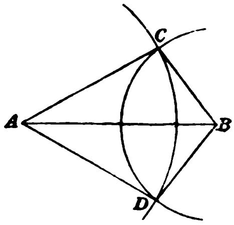
Make B A C=102°; A C=37; A C B=87°.

LESSON XVII.

TO CONSTRUCT A TRIGON OF WHICH THE THREE SIDES ARE KNOWN.

LET it be proposed to construct a triangle having its sides 48, 37, and 23 measures respectively.

If we draw a line A B of the length of 48 divisions of the scale, this may do for one side of the figure. Since the distance from A to the third corner is to be 37, that corner must be somewhere in the circumference of a circle described from A as a centre with a radius 37; but the same corner is to be at the distance 23 from B; wherefore it must be also in the circumference of a circle described from B with that radius. These circumferences cross each other at two points C on the one side of A B,



D on the other side; wherefore by joining A C, C B or by joining A D, D B we form a trigon of the required dimensions. The one of these could be folded over, as it were, upon the other.

Hence if the three sides of one trigon be equal respectively to those of another the figures are alike, and the angles of the one are equal to those of the other.

It is easily seen that if the longest line were more than the other two put together, the circles would not meet each other, and that there could be no trigon. Thus it is not possible to make a trigon with its sides 48, 27, 20; but if the longest of the three lines be less than the sum of the two others, the circles must overlap, so that the trigon may be made.

Exercise 1.

Construct the triangle A B C having A B = 97, B C = 97, C A = 97; bisect each of its sides and join the middle points with the opposite corners; measure those lines and all the angles.

Exercise 2.

Construct the trigon A B C having A B = 45, B C = 56, C A = 63, and make the figure for Exercise 2 of Lesson VI.

Exercise 3.

Make A B = 40, B C = 42, C A = 35, and repeat the construction for Exercise 3 of Lesson VI.

Exercise 4.

Make A B = 45, B C = 42, C A = 33, and repeat the construction for Exercise 4 of Lesson VI.

Exercise 5.

Construct a trigon with the sides 6, 25, and 29 parts.

Exercise 6.

Construct a trigon with 51, 40 and 13 for its three sides.

Exercise 7.

Construct a pentagon A B C D E, having A B = 16, B C = 25, C D = 15, D E = 37, E A = 17, A C = 39, and C E = 44.

Exercise 8.

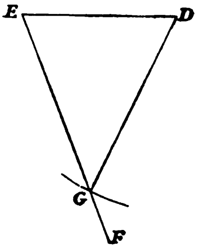
Construct a pentagon A B C D E, having A B = B C = C D = D E = E A = 55; A C = C E = 89; measure E B, B D, D A and all the angles.

LESSON XVIII.

To construct a trigon of which two sides and an angle opposite to one of them are known.

Let it be proposed to construct a triangle having two of its sides 46 and 35, the angle opposite to 46 being 68°.

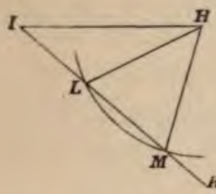
Since the angle of 68° is to be opposite to 46 it must be adjacent to 35, so that if we draw D E 35 long, and make the angle D E F of 68° we shall have part of the boundary of the figure. Now the distance from D to the third corner is to be 46, so that by sweeping an arc from the centre D with the radius 46, and noticing where it crosses the line E F we shall discover the third corner G.



In this example the angle is opposite the greater of the two sides: when it is to be opposite the less there arises a *certain difficulty.*

Let it be proposed, for example, to construct a trigon having two of its sides 43 and 31, the angle opposite 31 being 40°.

Having drawn H I equal to 43 and made H I K an angle of 40°, we sweep an arc from H with the radius 31.



This arc crosses the line I K twice, as at L and at M, and we do not know which of these to take. Indeed if H L and H M be both drawn, either of the triangles I H L, I H M, has the required dimensions. Hence it follows that two trigons may not be alike although two sides and an opposite angle in the one be equal respectively to the corresponding parts in the other. There is a doubt between two, and the proposition has a double answer.

It may happen that the length set down for the shorter side may be too little; the arc may not reach to the line I K, in which case there can be no such triangle.

Propositions like the preceding lessons in which something is proposed to be done are called *problems* from the Greek word *προβλήμα* which means something put forward; and the process of doing the thing is called the *solution* of the *problem*. Propositions again in which a *truth* or *law* is stated are called *theorems* from *θεώρημα* a spectacle, something to be looked at and studied. The following lesson is a theorem.

Exercise 1.

In the trigon ABC, the angle A B C is 118°, A B is twenty-eight, and A C is thirty-nine parts: construct the figure, measure B C and the other angles.

Exercise 2.

Make D E = 80 , D E F = 101°, D F = 109 .

Exercise 3.

Make G H = 73 , G H I = 81°, G I = 88 .

Exercise 4.

Make K L = 50 , K L M = 53°, K M = 41 .

Exercise 5.

Make N O = 97 , N O P = 42°, N P = 65 .

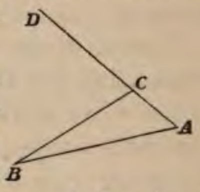
Exercise 6.

Make P Q = 78 , P Q R = 50°, P R = 58 .

LESSON XIX.

If one side of a trigon be produced, the outer or *exterior* angle is equal to both the farther angles put together.

Thus if the side A C be produced to D, the exterior angle B C D is equal to C B A and B A C put together.

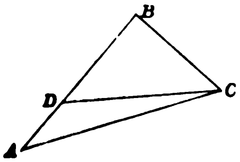


We perceive that it is so when we consider that the angle A C B together with B C D makes up half a turn; and that the three angles of the triangle, of which angles A C B is one, also make up half a turn, wherefore B C D must be just as much as the sum of C B A and B A C the remaining two angles.

LESSON XX.

The greater side of any trigon has the greater angle opposite to it.

Thus if the side A B be longer than B C, the angle A C B opposite to it is wider than B A C which is opposite to B C.



If we cut off from B A a portion B D equal to B C and join C D we form an isoskeles trigon having the angles B D C and D C B alike. Now A C B is greater than D C B by the part A C D, D C B is equal to B D C and B D C is greater than B A C by the same A C D, wherefore A C B is greater than B A C by twice A C D.

LESSON XXI.

The greater angle of a triangle has the longer side opposite to it.

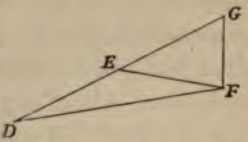
Thus if, in the preceding figure, the angle A C B be greater than B A C, the side A B is longer than B C.

For if we make the angle A C D equal to half the difference between the two angles A C B and B A C, the remaining angle D C B must be half the sum of the same two angles and must be equal to B D C; wherefore B C must be equal to B D and therefore shorter than B A.

LESSON XXII.

ANY two sides of a triangle are together longer than the third side.

For example, the two sides D E and E F put together are longer than D F. Let us continue D E until the part E G be equal to E F, then D G is as long as both D E and E F, or is their *sum;* join F G, then the angle E F G is equal to F G E, which may also be read F G D.



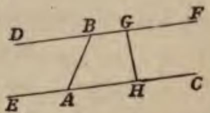
Now D F G is greater than E F G, wherefore it must be greater than F G D, and consequently the side D G opposite to it must be greater than D F.

LESSON XXIII.

ON PARALLEL LINES.

Two straight lines which make equal alternate angles with a third line cannot meet each other.

If at the two ends of the straight line A B we make equal angles but on different sides, or *alternately,* such as B A C and A B D, the lines D B and A C cannot meet, however far they may be continued.



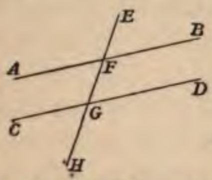
If these lines be produced in the direction A E, B F, the angles E A B and F B A must clearly be equal; and if we were to cut out the portion F B A C of paper, it would, when turned half round, just fit upon E A B D, however far the lines might be extended. Wherefore if the two lines A C, B F *were to meet at* some great distance, the others A E *would need to meet* at the same distance, that is to

two lines E A C, D B F would meet twice and therefore could not be straight.

Two straight lines as E C and D F which do not meet each other are said to be *parallel* from παράλληλος (along together).

If through the point B any other line be drawn, no matter how close to D B F, it would, if continued far enough come to meet E A C; or what is to say the same thing in other words, two parallels to the same line E C cannot cross each other. Also if any other straight line as G H be drawn from D F to E C, it makes equal alternate angles with them.

There is very frequent occasion to draw lines parallel to each other, so that we need some contrivance for doing this



easily and well. Now if there be two parallel lines A B, C D and if we draw across them a straight line E F G H, the angles B F H, D G H and their opposites A F E, C G E are all alike; as also the four angles E F B, E G D, A F H, C G H. Hence if the angle B F G were, so to speak, slid along the straight line E H, it would come into the position D G H.

If then we procure a piece of thin wood, with two sides A C, C B cut straight to any angle whatever, and if, hold-

ing a rule R R quite firmly on the paper, we apply thereto the side A C, and draw a line along C B; then, shifting the position of A C B by sliding it along the rule, if we draw a second line along C B, these lines must be parallel to each other. It is not usual to make the *sett* A C B of the shape shown in the figure; it is more *nvenient to* make it with three straight sides, with angles *table for* various purposes; very commonly one of the

angles is made 90° or 100^c; but for the mere drawing of parallel lines it does not matter what may be the angle.

If there be already drawn on the paper some lines to which we wish to make parallels, we first lay C B along that line, and then bring the rule R R up to the side A C. The rule is now to be held firmly and the *sett* A C B slid along it as far as may be needed. A very little practice suffices to make the learner expert at this operation.

Exercise 1.

Having made a triangle A B C, draw through A a line parallel to B C, and through C one parallel to B A, meeting the former in D; join B D crossing A C in E; through E draw E F parallel to A B, meeting B C in F, join A F crossing B E in G; draw G H parallel to A B, join A H cutting B D in I, make I K parallel to A B, and continue the operation in the same way. If this be rightly done B F is the half of B C; B H the third part; B K the fourth part, and so on.

Exercise 2.

Having made a convex tetragon A B C D and drawn its diagonals, through B and D draw parallels to A C continuing each both ways; through A and C draw parallels to B D, crossing the former parallels; the four-sided figure so formed is double of the tetragon.

Exercise 3.

Of the trigon A B C divide the side A B into three equal parts A D, D E, E B; draw E F and D G parallel to A C, *E I and D H* parallel to B C, then F H and G I should *parallel to B A,* and the three lines D G, E I, F H sh

pass through one point. The surface of A B C is thus divided into nine equal parts.

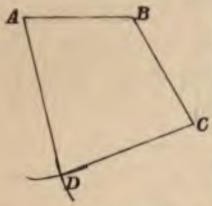
Exercise 4.

Make the same kind of construction with A B divided into five equal parts.

LESSON XXIV.

ON TETRAGONS.

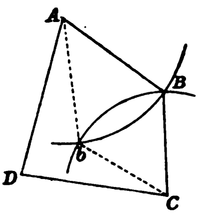
Every tetragon or four-cornered figure has four sides. The length of these four sides, however, does not *determine* or settle the shape of the figure. Thus if it were proposed



to construct a tetragon having its sides 23 , 27 , 31 , 35 we might draw A B, 23 ; and at any angle with it, B C, 27 ; then to find the place of the fourth corner we sweep an arc with the radius 31 from C as a centre, and cut that by another arc described from A as a centre with the radius 35. The tetragon or quadrangle A B C D would then have its four sides 23 , 27 , 31 , 35 as desired. But we might have made the angle A B C differently, and so we may construct a great many tetragons with the same lengths of sides, and yet not two of the figures alike.

In order to determine the figure completely we must have another measurement. If, for example the angle A B C were known, or if the distance A C were known, the *figure would* be determined; and thus *five* measurements

are needed to determine a four-cornered figure. We must not be too hasty, however, and conclude that five measurements are enough to fix the tetragon. Thus if it were proposed to construct A B C D having A B = 27, B C = 23, C D = 31, D A = 35 and the angle A D C = 81° = 90^c, we should begin by making A D 35, the angle A D C 81°, and D C 31. A, D and C are then certainly three corners of the figure. To find the fourth corner we describe two circles, one from A with the radius 27, the other from C with the radius 23. These two circles cut each other twice, at B and at *b*, so that we have two figures A B C D and A *b* C D each with the prescribed dimensions. Five *data* (*datum* a thing given) then are not always enough to determine a tetragon; we do not need a sixth measurement, but we must have some information to guide us in choosing between the two; thus if we be told that the angle at B is to be salient (projecting) or that it is to be re-entrant, we should know which of the two points to take.



Exercise 1.

Construct a convex tetragon A B C D having A B = 61, B C = 36, C D = 20, D A = 51, and A C = 65; measure B D.

Exercise 2.

Construct a tetragon E F G H having E F = 63, F G = 52, G H = 33, H E = 52, and E G = 25, and make a construction as in Exercise 2 of Lesson XXIII. Try t
out that the parallelogram is double of E F G H.

Exercise 3.

The tetragon I K L M has I K = 31, I K L = 75°, K L = 39, L M = 26, M I = 19, the angle at all being salient (less than 180°), construct it and measure K M.

Exercise 4.

Construct N O P Q with N O = 38, N O P = 65°, O P = 28, P Q = 13, Q N = 27, the angle at Q being re-entrant (greater than 180°). Measure O Q, and the angles at N, Q and P.

Exercise 5.

Construct R S T U with R S = 29, R S T = 82°, S T = 32, S T U = 109°, T U = 23. Measure S U, U R, R T, and the angles at U and R.

Exercise 6.

Construct V W X Y with Y V W = 99°, V W = 19, V W X = 102°, W X = 28, and W X Y = 93°. Measure V X, X Y, Y V.

Exercise 7.

Put together the measurements D A B = 112°, A B = 17, A B C = 115°, B C D = 60°, C D = 32.

LESSON XXV.

THE FOUR ANGLES OF A TETRAGON MAKE TOGETHER A WHOLE TURN.

We may easily be satisfied of this by using the illustration of Archimedes; or we may observe that every tetragon may be divided into two trigons whose six angles together just make up the four angles of the tetragon: now *the three angles* of each trigon make together half a turn,

wherefore the four angles of the tetragon must make up a whole turn.

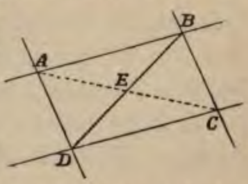
Hence if three angles of a tetragon be measured, the fourth angle may be calculated by adding the three together and subtracting the amount from 360° or 400^c.

In a tetragon a side is opposite to a side and an angle to an angle; thus A B is opposite to C D, and the angle C B A is opposite to the angle A D C; whereas in the trigon a side is opposite to an angle.

LESSON XXVI.

IF ONE PAIR OF PARALLEL LINES BE CROSSED BY ANOTHER PAIR, THE ENCLOSED TETRAGON HAS ITS OPPOSITE SIDES EQUAL TO EACH OTHER.

THUS if the parallel lines A B, D C be crossed by another pair of parallels B C, A D; the tetragon A B C D so formed has its opposite sides A B, C D alike, and also B C equal to A D. A figure of this kind is called a *parallelogram* (parallel lined), it is also called a *rhomboid.*



Having drawn one of the diagonals D B, we notice that the angle D B A is equal to its alternate B D C, and that A D B is equal to C B D, so that the two triangles into which the figure is divided must be alike in every way; the one turned half round would just fit upon the other; or if the whole tetragon were cut out, and turned half round it would fit again into the same place; the diag *BD then taking* up the position D B.

If the other diagonal A C be drawn, it would, when the figure is replaced, take up the position C A, and the point E at which the two diagonals cross each other would be again in the same place; wherefore A B=C D, B C=D A, A E=E C, B E=E D, and E is the centre point on which we may suppose the figure to have been turned. The opposite angles of a parallelogram are also alike.

EXERCISE 1.

Construct a parallelogram with an angle of 73 degrees, the containing sides being 37 and 43 . Draw and measure its diagonals.

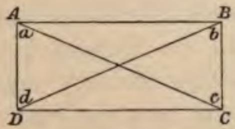
EXERCISE 2.

Make a rhomboid with an angle of 125°, the containing sides being 31 and 52 .

LESSON XXVII.

ON RECTANGLES AND SQUARES.

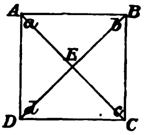
THE quarter of a turn is called a *right angle*, which is therefore 90 ancient or 100 modern degrees. An angle less than a quarter turn is called *acute* (from *acutus* sharp) and an angle between a quarter and half a turn is *obtuse* (blunt). Now all the angles of any tetragon make together one turn or four right angles, wherefore of the angles of a parallelogram if one, say that at A be acute the opposite angle at C must be acute also, and those at B and D must necessarily be obtuse. But if A be *exactly a right* angle, C must be also a quarter turn;



wherefore those at B and D must together make half a turn; but they are alike, wherefore each of them must be right; in other words, if one angle of a parallelogram be right, the three remaining angles are right also. A right-angled parallelogram is called a *rectangle;* the longer side is called its *length*, the shorter side its *breadth*.

The rectangle, like all parallelograms, may be turned half round and replaced; thus if the interior paper *a b c d* were separated from the outer part of the page, it might be turned half round so that C come to A, *d* to B, *a* to C and *b* to D. But it may also be reversed face for face and replaced, *c* being put to B, *d* to A, *b* to C and *a* to D; and still further it may then be turned half round. Thus a rectangle may be fitted into the same place in four different ways; two with one face up, two with the other face up; whereas a rhomboid having an acute angle cannot in general be reversed face for face. From this it is clear that the diagonals of a rectangle are of equal lengths.

If the length and breadth of a rectangle be made alike it is called a *square.* The square may be turned quarter round and replaced, thus *c* may be put to B, *d* to C, *a* to D and *b* to A; hence the square may be fitted into its place in eight different ways. We see also that the four angles round the point E are all alike so that each of them is a right angle.



EXERCISE 1.

Construct a rectangle 35 in length, by 12 in breadth, and measure its diagonals.

Exercise 2.

Construct a rectangle *under* 20 and 21 and measure its diagonal.

N.B. These two expressions are common, thus we say a board is 35 inches *by* 12 inches; or a rectangle *under* 35 and 12.

Exercise 3.

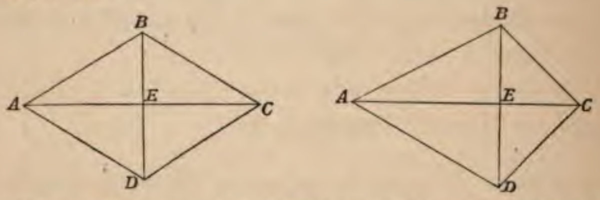
Make a square on twenty-nine parts, and measure its diagonal.

LESSON XXVIII.

ON THE RHOMBUS AND LOZENGE.

When the four sides of a parallelogram are all alike, the figure is called a *rhombus;* the word is Greek and very far-fetched; ῥομβος means *round,* and also denoted the shape of the leaden pellet used by slingers; this pellet was in the form of a double cone, therefore round when seen in one way, four-cornered when seen across the axis.

The rhombus may be turned half round, or it may be reversed face for face and replaced; like the rectangle it may be fitted in four positions. Hence its diagonals, though not of equal lengths, make four equal angles at the point E.



When a tetragon has its adjacent sides equal in pairs, *as when AB is* equal to AD, and CB to CD, we may

call it a *lozenge* (from the French *lozange*). This figure may be reversed face for face, the corners A and C returning to their former places; E D is equal to E B, and the angles at E are right angles.

It may be observed that the square belongs to all of these classes of figures; it is a parallelogram, a rectangle, a rhombus, and a lozenge. We may also notice that the middle of the sides of a parallelogram are the corners of another parallelogram; the middles of the sides of a rectangle are the corners of a rhombus; the middles of the sides of a rhombus are the corners of a rectangle; and lastly, those of the sides of a square are the corners of a square, as may be seen by examining the reversions of the figures. The middles of the sides of a lozenge are the corners of a rectangle; but the truth of this theorem is not quite so apparent.

Exercise 1.

Construct a rhombus having an angle of 54°, and each of its sides 35.

Exercise 2.

Construct a lozenge with A C = 37, A B = A D = 26, B C = D C = 15, and measure the cross diagonal B D.

Exercise 3.

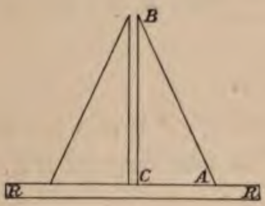
Construct a lozenge with A C = 40, A B = A D = 45, B C = D C = 13, and measure B D.

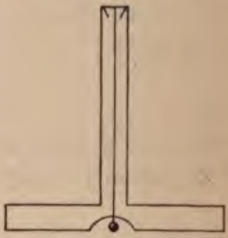
LESSON XXIX.

ON PERPENDICULARS.

The right angle is so often needed in business that instruments are constructed expressly for enabling us to draw it. For work on paper we use the *sett* described in Lesson XXIII., the angle B C A being made a right angle; and the side A B being made straight and inclined at some angle which may suit some particular purpose. The instrument in this form is called a *set-square.*

Our first business is to know whether the set-square be true or not. For this we apply it to a straight rule R R, and draw carefully a line along the free side. Then reversing it face for face we apply it again to the rule, bring the free side close to the line formerly drawn and examine whether it apply thereto, or we draw a second line close to the first and see whether these be parallel.

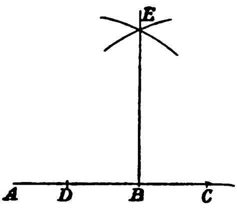




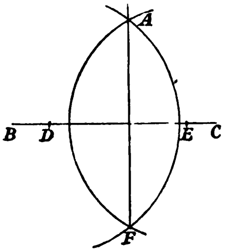
This is just the way in which a mason verifies his plummet; he places the instrument upon a level stone, changes its position face for face, and examines whether the cord to which the lead is tied come again to the same mark. Hence the name *perpendicular* is given to the sides of a right angle. Instead of this long Latin word *we might* use the shorter Greek one *normal* having

the same meaning; only in other branches of geometry this latter word is used with a slightly different signification.

We have sometimes to obtain perpendiculars by help of the compasses. Thus if it were required to raise at the point B a line perpendicular to the straight line A B C, we should measure on either side of B two equal distances B C, B D, and then from the points C and D as centres sweep arcs of equal circles crossing each other at E; the straight line B E is then perpendicular to A C.



Or if it were required of us to let fall from the point A a perpendicular to the line B C we should take in B C any convenient point D, and from D with the radius D A sweep an arc on the other side of the line. Then from some second centre E also in B C, we should draw another arc but with the radius E A, crossing the former arc at the point F, then the line A F is perpendicular to B C. For if D A, A E, E F, F D were drawn the figure would be a lozenge.



Exercise 1.

Construct a triangle with the sides 85, 66, 41, and from each corner let fall a perpendicular upon the opposite side, using the compass.

Exercise 2.

Construct a trigon with the sides 87, 74, 61, and dr— the three perpendiculars, using the set-square; mea the *perpendiculars.*

Exercise 3.

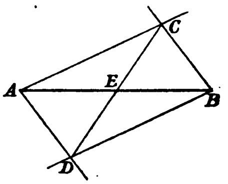
Construct the trigon with the sides 91, 73, 60; bisect the three angles, continuing the bisecting lines to meet the opposite sides; measure those lines and also the parts into which the sides are divided.

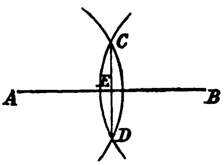
LESSON XXX.

TO BISECT OR HALVE A GIVEN STRAIGHT LINE.

In general the most convenient way of halving a straight line is to measure it by help of a good scale and to take the half of the number; thus if the line measure 58·6, its half is 29·3; and we may mark the middle point quite easily. We can hardly halve a line on the ground in any other way.

On paper we may bisect a line by drawing parallel lines. Thus if the middle of the line A B be wanted we may draw, by help of the set-square and a rule, two pair of parallel lines through A and B, crossing each other at C and D, then joining C D we obtain E the middle of A B.



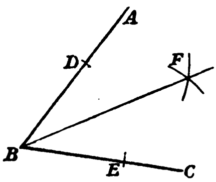


Or, using the pencil compass we may estimate a little more than the half line and, with that distance as a radius, may sweep two arcs of equal circles from A and B; joining the *cusps* C, D of these arcs, we obtain E the middle point.

Here if A C, C B, B D, D A were drawn, the figure would be a rhombus. On account of the number of operations neither of these processes is so trustworthy as the mode of trial, or that by help of the scale.

The bisection of an angle may be effected by the intersection of circles, or by help of the protractor.

Thus if it be required to draw a straight line which shall divide the angle A B C into two equal parts, we may measure the number of degrees, take the half of that number and make a mark accordingly. Or we may measure along B A and B C two equal distances such as B D, B E; from D and E sweep arcs of equal circles crossing at F, and join B F.

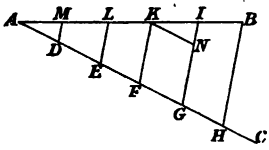


LESSON XXXI.

TO DIVIDE A STRAIGHT LINE INTO EQUAL PARTS BY HELP OF PARALLEL LINES.

Let it be required to divide the straight line A B into five equal parts.

From A, one of the ends, let us draw at any angle a line A C; then assuming some distance, let it be laid off five times along A C, as A D, D E, E F, F G, G H. Join now H B and through G, F, E and D draw parallels to H B; these parallels cut B A into five equal parts at I, K, L, and M. In order to show that these parts are equal, we *may draw K N parallel* to F G, thus forming a parallelo



gram F G N K; K N then is equal to F G. But on account of the parallelism of G I and D M, the angle K I N is equal to A M D; and because K N is parallel to A C, the angle I K N is equal to M A D, so that the trigon K I N is a copy of A M D, and K I is equal to A M. In the same way it may be shown that each of M L, L K, I B is equal to M A, and thus the line A B is equally divided into five parts.

Thus we may conclude generally that if a set of parallels divide one straight line into equal parts, they divide any other straight line drawn across them also equally.

This theorem contains the principle of all linear dividing engines.

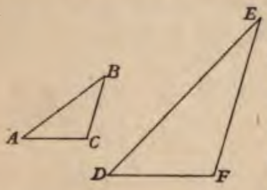
EXERCISE.

Divide a line of 3 inches into eleven equal parts.

LESSON XXXII.

ON RATIOS AND PROPORTIONS.

If two triangles A B C, D E F have their angles alike we say that they have the same shape; they may differ very much in size. The side A B is longer than A C, and in the other trigon D E is longer than D F; *comparatively* as much longer; in reality the difference between D E and D F is, in the present instance, much more than the difference between A B and A C. The comparison of two magnitudes of the same kind leads us to *form an idea of* what we call their *ratio*: if A B were



twice as long as A C, D E would be twice as long as D F; A B is not quite the double of A C, and D E is not quite the double of D F. If A C were divided into three equal parts A B would contain about five of those; and if D F were also divided into three parts D E would be about five of these.

Thus the comparison of two magnitudes is always made by help of numbers.

We divide one of the magnitudes into equal parts, or we imagine the division, and then consider how many of these parts go to make up the other of the magnitudes. If, for example, on dividing A B into five parts we find that A C contains three of them, we say that A C is three fifth parts of A B, that A B is five third parts of A C, that the ratio of A B to A C is that of five to three, or that the ratio of A C to A B is the ratio of three to five. These different statements are in truth only different ways of making the same statement; we abbreviate them thus

$$AC = \tfrac{3}{5}AB;\ AB = \tfrac{5}{3}AC;\ AB:AC::5:3;\ AC:AB::3:5.$$

Now, in the above trigons, if A B be five-thirds of A C, D E is five-thirds of D F; so we say that the ratio of A B to A C is the same as the ratio of D E to D F, and we write this as under,

$$AB : AC :: DE : DF$$

$$\text{or } AC : AB :: DF : DE.$$

Ratios occur in every branch of business. Thus the partners in a mercantile affair divide the profit in proportion to their shares of the capital; the r[illegible] whole capital to the whole profit being also the [illegible] *partner's share* of the capital to his share of th[illegible]

is very common to state the rate of profit as so much per hundred of the capital; thus a banking company may declare *dividends* at 23 per cent.; that is 23 for every hundred of capital, the ratio of the capital to the profit being in such case 100 : 23.

If we dissolve a quantity of salt in water, and stir the whole till it be uniformly mixed; the ratio of the weight of salt to the weight of water in a single spoonful of the solution, is the same as that of the total weight of the salt to the total weight of the water.

Thus *ratio* does not concern the absolute magnitudes, it has to do only with the comparative magnitudes. Two very small objects may have the same ratio as two very great ones. This idea runs through all language, thus we speak of a large fly, or of a small house; of a fly larger than flies usually are; of a house small in comparison with other houses.

It is more easy to operate on straight lines than on any other kind of magnitude: wherefore we shall exemplify what we have to say on this subject by help of them; or, as it may otherwise be stated, we shall represent magnitudes in general by straight lines, while treating of ratios.

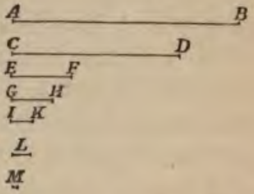
LESSON XXXIII.

TO EXPRESS BY NUMBERS THE RATIO OF TWO GIVEN MAGNITUDES OF THE SAME KIND.

Let the two proposed magnitudes, two angles, two weights, two money values, be represented by the straight lines A B, C D; and let it be required of us to find their ratio in *numbers.* We should naturally say, measure them with

some scale, and then the numbers will give what is wanted; but here we observe that neither of the two lines may be exactly so many divisions, so that the method, though very convenient for general purposes, is, strictly speaking, inapplicable, unless we can find a scale on which both may be measured. The commercial plan is to divide one of them say A B into 100 equal parts and to measure C D by those parts; in the example before us C D is nearly 73 of them, that is C D is about 73 per cent. of A B; but then the hundredth part of A B may not go *exactly* in C D.

Thus, in order to get the true numerical expression for the ratio, we must find out into what number of parts A B must be divided so that C D may contain so many of them without error. Whichever way we take it, the problem comes to be this, "to find some line which shall be contained exactly in A B and also exactly in C D," or as it is usually



expressed, "to find the *common measure* of A B and C D." To shorten our language we shall call this unknown common measure M. It is quite clear that if M may be stepped exactly from A to B, and also exactly from C to D, the excess of A B above C D must also contain M so many times without remainder; wherefore if we make E F equal to the difference between A B and C D, M must be a measure of E F; and now we must seek for a measure common to C D and E F.

Proceeding in the same way we take E F from C D, the remainder is more than E F, we therefore again deduct it and find that C D less twice E F is G H; the common measure M then must step exactly from G to H;
be common to E F and G H.

Continuing the same process we take G H from E F and find the remainder I K; taking this I K from G H there remains L, which again taken from I K leaves the small excess M, so small that we find it difficult to proceed farther; M, as far as we can make out, goes six times in L. If it really do go exactly six times it is the measure common to all the lines I K, G H, E F, C D and A B. On account of the errors incidental to all such measurements, we can only say that M is, as nearly as can be found, the required common measure.

Having now got M which exactly measures both C D and A B, we have to count how many times it goes in each of them. To do this we observe that $L = 6\,M$, that $IK = L + M = 7\,M$; again $GH = IK + L = 7\,M + 6\,M = 13\,M$; $EF = GH + IK = 13\,M + 7\,M = 20\,M$; $CD = 2\,EF + GH = 40\,M + 13\,M = 53\,M$; and lastly $AB = CD + EF = 53\,M + 20\,M = 73\,M$; so that $AB : CD :: 73 : 53$.

Drawing two straight lines at random, the learner should practise this operation until he be satisfied that he completely understands it.

It may happen, indeed it often does happen, that the two magnitudes have no common measure; in such cases the ratio cannot be accurately expressed in numbers; but in all cases we may find a numerical expression sufficiently near for any actual purpose. The case of incommensurables will be discussed in a future lesson.

A convenient arrangement of the work may be used as under. Using a single letter to denote each line or other magnitude, let us suppose that, in seeking the ratio of a to b, we find that b goes twice in a with a remainder c, that c goes once in b with d over; that d goes five times

in c with e over, and so on; the results of these trials may be written

$$\begin{aligned} a &= 2\,b + c \\ b &= 1\,c + d \\ c &= 5\,d + e \\ d &= 3\,e + f \\ e &= 1\,f + g \\ f &= 4\,g + h \\ g &= 3\,h \end{aligned}$$

in which case h is the common measure. Beginning with the last and proceeding backwards we find

$$\begin{aligned} f &= 4\,g + h = 12\,h + h = 13\,h \\ e &= 1\,f + g = 13\,h + 3\,h = 16\,h \\ d &= 3\,e + f = 48\,h + 13\,h = 61\,h \\ c &= 5\,d + e = 305\,h + 16\,h = 321\,h \\ b &= 1\,c + d = 321\,h + 61\,h = 382\,h \\ a &= 2\,b + c = 764\,h + 321\,h = 1085\,h \end{aligned}$$

so that $a : b :: 1085 : 382$.

Space and time may be saved by writing the successive results at once opposite the lines, as in the following example,

$$\begin{aligned} a &= 4\,b + e = 252\,f \\ b &= 3\,c + d = 59\,f \\ c &= 1\,d + e = 16\,f \\ d &= 2\,e + f = 11\,f \\ e &= 5\,f \quad\;\; = 5\,f \end{aligned}$$

Exercise 1.

Make a square and find the ratio of its diagonal ' side.

Exercise 2.

Divide the circumference of a circle into five equal parts, join the points of division so as to form a pentagon; draw also one of the diagonals and find the ratio of that diagonal to the side of the pentagon.

Exercise 3.

Make an equilateral trigon, draw a perpendicular from one corner to the opposite side and find its ratio to one of the sides.

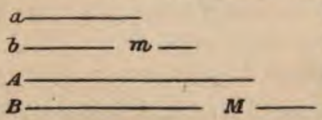
Exercise 4.

Divide the circumference of a circle into seven equal parts; draw the sides and diagonals of the heptagon. Find the ratio of the longer diagonal to the side; find also the ratio of the shorter diagonal to the same side, and thence compute the ratio of the one diagonal to the other: verify the result by trial.

LESSON XXXIV.

THE RATIO OF TWO MAGNITUDES IS THE SAME AS THAT OF THEIR DOUBLES, TRIPLES OR OTHER EQUIMULTIPLES.

Let a and b be two magnitudes; A and B their doubles; let also m be the common measure of a and b; and make



M double of m; then it is clear that M goes in A just as often as m goes in a, and as often in B as m goes in b; wherefore the same numbers which *express the ratio* of a to b express also that of A to B.

The identity of these ratios is expressed by the ordinary notation

$$a : b :: A : B$$

which is called a proportion; A and B being said to be proportional to a and b. The same statement may be made in the form

$$b : a :: B : A.$$

If instead of the doubles we were to take any other multiples, the same reasoning would hold good, thus we may write

$$a : b :: 5a : 5b$$

or if n stand for any number whatever

$$a : b :: na : nb.$$

Of course the same theorem holds good of the halves, third parts and other *submultiples* of two quantities; for if A and B be the doubles of a and b, these again are the halves of A and B. That is in general $a : b :: \frac{1}{n}a : \frac{1}{n}b$; and consequently if r and s stand for any two numbers,

$$a : b :: \frac{r}{s}a : \frac{r}{s}b.$$

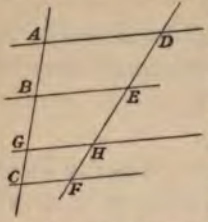
LESSON XXXV.

PARALLELS DIVIDE ALL INTERSECTED LINES PROPORTIONALLY.

If three parallel straight lines be crossed by the lines A B C, D E F, these latter are cut proportionally, that is to say, the ratio of A B to B C is the same with the ratio of D E to E F.

In order to find the ratio of A B to B C, we take A B which happens to be the less, out of B C; it goes *with G C over.* Through G draw G H parallel to

and cutting E F in H, then, from what was shown in Lesson XXXI., E H is equal to D E. Thus, however



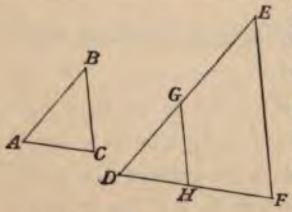
often A B may go in B C, just so often does D E go in E F, and if there be a remainder on the one line there must be a corresponding remainder on the other. Wherefore the numbers expressing the ratio of A B to B C must also express that of D E to E F.

We may therefore write A B : B C :: D E : E F, and we may alternately say that the ratio of A B to D E is the same with that of B C to E F. For if M be the common measure of A B and B C, while N is the common measure of D E and E F, we must have M : N :: A B : D E, because A B and D E are equimultiples of M and N. In the same way M : N :: B C : E F, and therefore A B : D E :: B C : E F.

LESSON XXXVI.

IF THE ANGLES OF ONE TRIGON BE EQUAL TO THOSE OF ANOTHER, THE SIDES ARE PROPORTIONAL.

Thus if the angles B A C and C B A be equal respectively to E D F and F E D, the sides of the triangle A B C are



proportional to those of D E F; that is to say the ratios of A B to D E of B C to E F and of C A to F D are all the same. For if we measure along D E the distance D G equal to A B and draw G H parallel to E F, we shall thereby form a trigon D G H equal to A B C. Now *D G* : *D E* :: D H : D F according to what was shown

in the preceding lesson, wherefore since D G and D H are copies of A B and A C, A B : D E :: A C : D F. The trigon A B C is in such a case said to be similar to D E F.

If we provide two scales of equal parts such that A B shall measure as many parts on the one as D E measures on the other; then the measurements of B C and C A on the former scale will correspond with those of E F and F D on the latter.

Hence if we construct two trigons with the same numbers of parts but measured from different scales, their angles will be alike; they will have the same shape or be *similar* to each other, although they differ in size.

EXERCISE.

Construct a trigon A B C having A B = 53, B C = 59, C A = 71, measure its angles. Make a second trigon having D E = 37, having also the angle E D F = B A C, and F E D = C B A. Compute now the lengths of E F and D F, and verify the computations by measurement.

LESSON XXXVII.

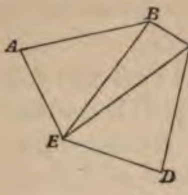
TO CONSTRUCT A FIGURE SIMILAR TO A GIVEN FIGURE AND IN A GIVEN PROPORTION.

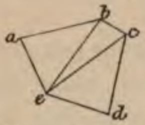
THE pentagon A B C D E being given, it is proposed to make another similar to it and on a scale of two-thirds.

In this case if the scale for A B C D have twenty divisions to the inch, the scale for the new figure must have thirty.

Having divided the pentagon into trigons by dra

two diagonals as B E, E C; make somewhere on paper a line $a b$ two-thirds of A B, that is measuring as many divisions on scale 30 as A B measures on scale 20; measure now with the protractor all the angles in the figure and make $b a e$, $e b a$ equal to B A E, E B A, thus obtaining the trigon $a b e$ similar to A B E. At e and b make angles equal to those at E and B, so getting $b c e$ similar to B C E, and thus proceed until the figure be completed.





Otherwise we may measure all the lines in the given figure by scale 20, and with the corresponding numbers of parts from scale 30 construct the three triangles.

In the former process the proportionality of the lines results from the equality of the angles; in the latter process the equality of the angles comes from the proportionality of the lines.

Exercise.

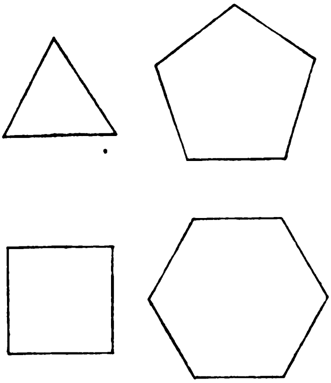
Construct a pentagon $a b c d e$ similar to that of Exercise 7, Lesson XVII., and having $a b = 20$.

LESSON XXXVIII.

ON REGULAR POLYGONS.

If the circumference of a circle be divided into equal parts, and if the chords of these parts be drawn, a polygon will be formed; which, if cut out, may be turned partly round and replaced. All its sides are alike and all its angles are alike; such a polygon is said to be *regular*.

A regular figure of three sides, or an equilateral trigon, has each of its angles the sixth part of a turn, that is 60 ancient or $66\frac{2}{3}$ modern degrees; for all three together make half a turn. A regular figure of four sides, or a square, has each angle one-fourth of a turn, that is 90 ancient or 100 modern degrees, as we have already seen. All the angles of a pentagon make together three half turns, that is half a turn for each of the three triangles composing it; wherefore each angle of a regular pentagon must be three-tenths of a turn, that is $108^\circ = 120^c$.



A figure of six sides is composed of four trigons, wherefore all its angles make together two turns; so that the angle of a regular hexagon must be just one third part of a turn or $120^\circ = 133\frac{1}{3}^c$.

In the same way we may compute the angles of regular figures of a greater number of sides. The angle of a regular heptagon, for example, is $\frac{5}{14}$ of a turn.

In general if n be put for the number of corners, $n-2$ is the number of trigons composing the figure, and therefore $\frac{n-2}{2}$ the number of turns equal to the sum of all its angles. Taking the nth part of this to get the value of one of them when they are all alike, the formula

$$\frac{n-2}{2n}$$

gives it in fractions of a turn; this may otherwise be written $\frac{1}{2} - \frac{1}{n}$.

EXERCISE 1.

Cut, in cardboard, a number of equilateral trigons and put them together.

EXERCISE 2.

Cut a number of squares to serve as paving tiles.

EXERCISE 3.

Cut a few regular pentagons of one size.

EXERCISE 4.

Cut a number of regular hexagons, and put them together.

EXERCISE 5.

Cut three regular heptagons of one size.

LESSON XXXIX.

ON THE MEASUREMENT OF SURFACE.

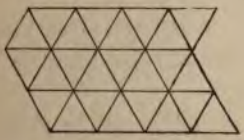
IN order to measure a line we take some known length, an inch, a foot, a mile, as the *unit* of length and, adding this unit again and again to itself, try how many times it must be repeated to make up the line which is to be measured. So it was in angles; the real or absolute unit of angle is the whole turn, but for convenience we divided this into degrees and *took the degree* as the unit: an angle was measured by

laying one degree to another (on the protractor) until the proposed angle was made up. So it is, and so it must be in the measurement of all kinds of measureable magnitudes. In order to measure *weight* or to weigh we take some known weight, as an ounce, for the unit, and add ounce to ounce until we equipoise the thing to be weighed.

When we have to measure *surface*, we must take some known bit of surface and repeat this until the whole figure which is to be measured be covered up. Here we meet with a difficulty: the unit must not only be known in *size*, but its *shape* must be such that it may be repeated side by side each way.

There must be some connection between the unit of length and the unit of surface; the obvious connection is that the unit of surface should be some regular figure with an inch or other linear unit for one of its dimensions; and therefore we must seek for some regular figure capable of being repeated side by side.

Beginning with the regular three-sided figure we observe that its angle is the sixth part of a complete turn, so that six trigons may be placed round a point without leaving any open space; hence if we cut a great many equilateral trigons in paper we may pack them side by side; they may be used for covering a floor or a table.



The angle of the regular four-sided figure is the fourth part of a turn, so that four such figures may be placed a point. *Hence* squares may be used for covering

The regular five-sided figure has an angle of three tenth parts of a turn; hence three pentagons may be placed round a point, but with an open or uncovered angle of one tenth part of a turn, that is of $36° = 40^c$. The regular pentagon then cannot be taken as the unit of surface.

The regular six-sided figure has an angle of one third part of a turn; three hexagons may be placed round a point without any gap, and therefore hexagons may be used for paving or covering surface.

The angles of all other regular figures are more than the third part but less than the half of a turn. Three of them cannot be placed round a point and two of them must always leave an open; hence the only regular figures which can be used as the surface-unit are the trigon, the tetragon and the hexagon.

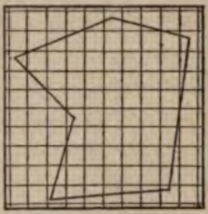
We shall afterwards have to measure *bulk* or *solidity*; now there are only five kinds of regular solids, and of these only one can be used for building or filling up space. Each of the remaining four would leave opens between. The cube has six square faces, and eight cubes may be placed around a point leaving no open; wherefore as the cube must be used in the measurement of bulk, the square must be taken as the unit of surface.

If, then, we measure lines in inches we must measure surface in square inches; if we use feet for lines we must use square feet in surface.

Although we may place square inches side by side so as to cover up surface without any openings between the squares, we cannot place them so as to fill up any figure having oblique angles, that is angles not right angles. There are always corners projecting beyond the boundary,

or portions awanting. The measurement of surface is thus somewhat troublesome.

We may rule upon a piece of plate glass or on transparent paper two series of parallel lines forming little squares, and placing it over any figure which is to be measured, may count the number of complete squares contained within it; and then estimate the fractional pieces round the sides. To save part of this latter trouble we may put one of the ruled lines over a straight side of the figure; also, to enable us to count in square inches, we should make the interval between the parallel lines some fraction, say the tenth part of an inch. The surface of a figure is often called its *area;* the word *area* in Latin means a dried up or waste place, and thence an open or unoccupied place.



LESSON XL.

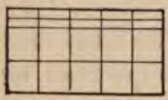
TO MEASURE A RECTANGLE.

When each side of a rectangle contains the linear unit exactly, there is no difficulty in estimating its surface; we have only to rule it, or to fancy it to be ruled into squares. Thus, of the adjoining rectangle, if the length contain eleven units and the breadth six units, the surface, clearly, contains six times eleven, that is sixty-six square units. There are six rows of eleven squares each, or, taken cross there *are eleven* rows of six squares each.

It is very common to state that "the product of the length by the breadth of a rectangle gives the area," and this may be a convenient way of putting the matter; but the student must be careful not to be misled by it. We cannot multiply 11 inches by 6 inches; a magnitude cannot be a multiplier. We may multiply (that is take many times) a line by a number; but to speak of multiplying a length by a length is to talk nonsense. The multiplicand (that which is multiplied) may be a number or may be a quantity, the multiplier can only be a number, and the result or *product* is necessarily a quantity of the same kind as the multiplicand.

In the present case the multiplicand is the square unit and there are two multipliers, namely, the *number* of the units in the length and the *number* of the units in the breadth. The surface-unit multiplied by 11 gives us eleven square units for one row, and that multiplied by 6 (the number of the rows) gives sixty-six square units; the multiplicand is surface, and the product is a magnitude of the same kind, it is surface also.

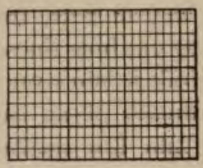
The computation of the area of a rectangle whose dimensions are in integer numbers is easy: let us now examine the case in which one of the dimensions is fractional, as when the length is 5 inches and the breadth $2\frac{2}{3}$ inches.



Here if we draw parallels dividing the length into inches, and intersect these by parallels at each inch of the breadth as well as at each part of the remaining two-thirds of an inch, we perceive that the area consists of ten undivided square inches and of ten portions each of which is the third part of a square inch; these *ten portions* make as much as three square inches and

one third part, wherefore the whole rectangle contains $13\frac{1}{3}$ square inches.

In order to obtain the area of a rectangle $3\frac{2}{7}$ inches long, $2\frac{3}{5}$ inches broad, we may construct one 4 inches long by 3 inches broad, and divide it into square inches; then dividing the last inch of the length into seven parts and the last inch of the breadth into five parts we draw parallels through the points of section and by their help determine the required rectangle as is shown by the strong lines in the figure.



Here we have $3\times2=6$ entire square inches; along the lower side there are $3\times3=9$ pieces each of them one fifth part of a square inch, along the side there are $2\times2=4$ pieces each one seventh part of a square inch; and lastly, in the very corner, there are $2\times3=6$ small rectangles of which thirty-five go to make up the superficial unit; hence the area is composed of four parts

$$6+\frac{9}{5}+\frac{4}{7}+\frac{6}{35} \text{ or } 6+1\frac{28}{35}+\frac{20}{35}+\frac{6}{35}=8\frac{19}{35}.$$

Or we may proceed thus; having made the rectangle as in the second figure, we divide the length into inches and seventh parts of an inch, in all twenty-three seventh parts, drawing parallels through the points of section and we divide the breadth into inches and fifth parts an *inch, drawing* again parallels. In this way the wh

rectangle is divided into $23 \times 13 = 299$ small rectangles of which 7×5 are contained in the square unit; so that the area is $\frac{299}{35} = 8\frac{19}{35}$. The arithmetical process here is to multiply together the two fractions $\frac{23}{7}$ and $\frac{13}{5}$.

Now a fraction necessarily represents a quantity, and therefore, strictly speaking, one fraction cannot be multiplied by another. The phrase "multiplication of two fractions" though convenient is apt to give rise to indistinct notions. In the above operation we have in reality four distinct steps, two divisions and two multiplications. Setting out from the square inch or other unit of surface, we divide it into seven equal parts each one inch long and one-seventh of an inch broad; next we subdivide that part into five smaller rectangles each one-fifth of an inch long and one-seventh of an inch broad. This small rectangle, which is the thirty-fifth part of a square inch, now taken as the temporary unit, is multiplied by 23 to make up one of the long rows, and this again by 13 to make up the whole rectangle.

EXERCISE 1.

How many square inches are in the surface of a rectangle 23 inches long by 17 broad?

EXERCISE 2.

How many square yards of carpeting will cover a floor 8 yards long by 5 yards broad?

EXERCISE 3.

A rectangle is to contain 133 square inches, what must *be its length* when the breadth is 7 inches?

Exercise 4.

Required the area of a rectangle $27\frac{1}{2}$ inches long by 14 inches broad.

Exercise 5.

Construct a rectangle $3\frac{1}{2}$ inches long by $2\frac{1}{2}$ broad, compute the area and show the details on the figure.

Exercise 6.

Required the area of a rectangle $15\frac{2}{5}$ units long by $9\frac{2}{7}$ units broad.

Exercise 7.

A board is $7\frac{3}{16}$ inches broad, by 3 feet $5\frac{1}{4}$ inches long, required its surface.

Exercise 8.

What is the area of a rectangle $\frac{11}{13}$ of an inch long by $\frac{5}{17}$ of an inch broad?

Exercise 9.

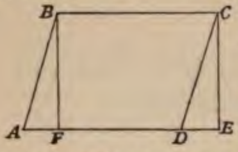
Required the surface of a board 43·27 long by 7·41 broad.

LESSON XLI.

TO MEASURE A RHOMBOID.

When the angle of a parallelogram is not right, the surface cannot be covered by squares. We cannot measure it directly, yet by a very simple expedient we may obtain the measurement. Thus if the rhomboid be A B C D, having an acute angle at A, we may draw from B a perpendicular B F to A D, thus cutting [illegible] *trigon A B F.* By drawing through C the [illegible]

parallel to B F and meeting the prolongation of A D in E, we add the triangle D C E which is equal to A B F, and there is as much surface in the rectangle F B C E as there is in the rhomboid A B C D.



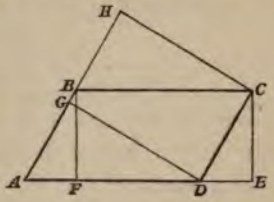
When two figures containing like quantities of surface have not the same shape, they can hardly be said to be equal to each other; it is needed, for clearness sake, that we distinguish between complete equality and this inferior degree of it. Modern geometers say of two figures that have equal extents of surface, without reference to their shapes, that they are *equivalent,* that is of equal value (as it were at so much per square inch). The rectangle F B C E is, then, equivalent to the rhomboid A B C D.

Sometimes we call one side the *base* of a rhomboid, and the perpendicular drawn to it from the opposite side the *altitude;* thus, if D A be taken as the base, B F is the *altitude* of the rhomboid A B C D. Hence it appears that a rhomboid is equivalent to a rectangle having the same base and the same altitude, and hence also rhomboids on equal bases and having equal altitudes are equivalent to each other.

Thus in order to compute the area of a rhomboid we measure its base and we measure its altitude (or breadth square across) and multiply together the numbers of units in each, the result being the number of square units in the surface of the rhomboid.

Any side may be assumed as the base of a rhomboid; if A B were taken as the base, the perpendicular D G would be the corresponding altitude. Thus we may roceed in either of two ways, obtaining in the one

F B C E, in the other G H C D; which rectangles being each equivalent to the rhomboid are equivalent to each other. In measuring a rhomboid it is prudent to use both processes and to compare the results.



We see here that the rectangle F B C E is equivalent to the rectangle G H C D: this circumstance deserves special attention.

If we compare the trigon A D G with A B F, we find the angle at A common to both, and the right angle A G D equal to A F B. These two triangles then are alike in shape, and their sides are proportional, whence

$$AD : AB :: DG : BF.$$

In this proportion the *extreme* terms (the first and the fourth) are the length and the breadth of one of the rectangles, while the *mean* terms (the second and the third) are the length and the breadth of the other rectangle; whence the important *theorem*:—

If four straight lines be proportional, the rectangle under the extremes is equivalent to the rectangle under the means.

Exercise 1.

Construct a rhomboid with an angle of 64°, the containing sides being 89 and 125; and compute its area.

Exercise 2.

Required the area of a rhomboid having its sides 75 137 with the contained angle 40°.

Exercise 3.

The sides of a rhomboid are 149 and 143, while their contained angle is 70°, required the area.

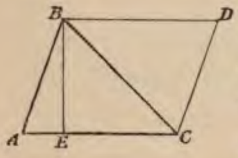
Exercise 4.

The two sides of a rhomboid are 156 and 84, the altitude on the former is 63., compute the altitude on the latter; construct also the figure.

LESSON XLII.

TO MEASURE THE SURFACE OF A TRIGON.

If we draw B D parallel to A C, and C D parallel to A B, we form a rhomboid just double of the triangle A B C. Drawing now B E perpendicular to A C we perceive that the rhomboid A B D C is equivalent to the rectangle under A C and B E; wherefore the area of a trigon is equivalent to the half of a rectangle under its base (A C) and its altitude (B E).



Since we may regard any one of the three sides as the base, and the perpendicular upon it from the opposite corner as the altitude, there are three modes of measuring the area; the results of the three ought to be the same.

If one of the angles be obtuse, the altitudes or perpendiculars let fall upon the sides of that angle from the opposite corners are outside of the figure. This, however, *does not affect* the truth of the above theorem.

Exercise 1.

The three sides of a triangle are 33, 41 and 58, construct it and compute its surface in the three ways.

Exercise 2.

What is the area of a trigon having its sides 6, 25 and 29?

Exercise 3.

The three sides being 51, 52, 53, required the area.

Exercise 4.

What is the surface of an equilateral trigon on the base 52?

Exercise 5.

Required the surface of a trigon whose sides are 65, 76 and 87.

Exercise 6.

Required the surface of a trigon whose sides are 9, 65 and 70.

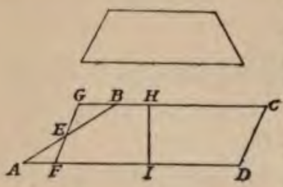
LESSON XLIII.

TO COMPUTE THE AREA OF A TRAPEZOID.

A four-sided figure having two of its sides parallel and the other two equally inclined inwards is called a *trapezium* from the Greek *τράπεζα* a table. The table was formed of boards laid on two tressels, thus having the outline of a trapezium. When the non-parallel sides are irregularly placed, as in the figure A B C D, it is called a trapezoid (somewhat like a trapezium).

If we halve one of the oblique sides, as AB, in E, and

through the middle point draw F E G parallel to the other oblique side D E, we shall form a rhomboid equivalent to the trapezoid, because the surface A E F cut off by F E is equal to B E G. The base F D of this rhomboid is as much shorter than A D as it is longer than B C; it is then half their sum, or as it is called the *arithmetical mean* between the parallel sides of the trapezoid.



Now F G C D is equivalent to a rectangle under F D and the altitude H I; wherefore the area of any trapezoid is equivalent to the rectangle under its altitude and half the sum of its parallel sides.

EXERCISE 1.

Construct a trapezoid on a base of 120 with an altitude 48, while the oblique sides, both sloping inwards, are 73 and 60. Compute its area.

EXERCISE 2.

A trapezium on a base 70, with an altitude 45, has each of its ends sloping inwards 53 long; construct it and compute the area.

EXERCISE 3.

At the ends of a line 60 long, two perpendiculars are drawn 53 and 64 respectively; their ends are joined; required the area of the trapezoid so formed.

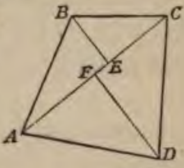
EXERCISE 4.

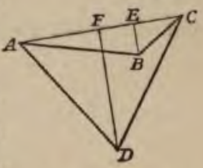
Construct the figure A B C D having A B = 37, A B C = 71°, B C = 37, B C D = 109°, and C D = 25; and compute its area.

LESSON XLIV.

TO MEASURE THE SURFACE OF A TETRAGON.

When it is required to ascertain the area of a convex four-sided figure A B C D, we may draw A C one of its diagonals, and on it let fall the perpendiculars B E, D F from the other corners. The trigon A B C is equivalent to half the rectangle under A C and B E, while C D A is equivalent to half the rectangle under the same A C and D F, wherefore the whole area A B C D is equivalent to a rectangle under A C and half the sum of the perpendiculars B E and D F.





But when there is a re-entrant angle A B C, the line A C lies without the figure, which figure is then the difference between the triangle C D A and C B A. In this case A B C D is equivalent to the rectangle under A C and half the difference of the perpendiculars D F and B E.

We might have drawn the diagonal B D and upon it have let fall perpendiculars from A and C; so that the comparison of the two results may serve to show the degree of accuracy of the measurements.

Exercise 1.

Construct the tetragon A B C D with A B = 15, B C = 37, C D = 39, D A = 17, A C = 44, and compute the area.

Exercise 2.

Construct the tetragon D E F G, having the angle at G re-entrant, with the measurements D E = 34, E F = 25, F G = 17, G D = 28, and D F = 39, and compute its area.

EXERCISE 3.

The dimensions of H I K L are H I = 63, I K = 16, K L = 56, L H = 33, and H K = 65; construct the tetragon and compute its surface.

EXERCISE 4.

Construct the convex tetragon M N O P with M N = 46, N O = 31, O P = 17, P M = 38, and M O = 45, and compute its area.

EXERCISE 5.

Construct Q R S T, having T re-entrant, with the measurements Q R = 121, R S = 100, S T = 40, T Q = 79, and Q S = 90, and compute its surface.

LESSON XLV.

TO MEASURE THE SURFACE OF ANY POLYGON.

BY drawing lines across from corner to corner we may always divide the surface of a polygon into tetragons and trigons; and by measuring the area of these and taking their sum, we obtain the total area of the figure.

EXERCISE 1.

Construct the convex pentagon A B C D E to the measurements B E = 65, E A = 12, A B = 55, B C = 57, C E = 68, C D = 43, D E = 61, and compute its area.

EXERCISE 2.

Construct the convex hexagon F G H I K L to the dimensions L F = 44, F G = 35, G H = 65, H I = 29, I K = 51, K L = 40, L G = 75, G I = 68, I L = 77, and compute its surface.

Exercise 3.

Construct the convex heptagon A B C D E F G having A B = B C = C D = D E = E F = F G = G A = 81, and A E = E B = B F = F C = 182, and compute its area.

LESSON XLVI.

ON RECTANGLES.

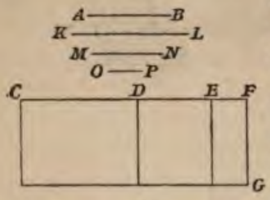
Rectangular surfaces occur so often in business that it is exceedingly useful to know their properties and relations. We have already seen that the number expressing the area of a rectangle is the product of the numbers which express its length and its breadth; hence the doctrine of rectangles must be related to that of products in arithmetic. Now if the letters a and b stand for two numbers we are accustomed to write $a \times b$, $a . b$ or simply $a b$ for their product; so if two lines P Q and V W be the one the length and the other the breadth of a rectangle we denote the *area* of that rectangle by the symbol P Q . W V, and we call it the rectangle *under* P Q and W V, or the rectangle *contained* by P Q and W V; those forms of speech, though somewhat meaningless, have got into general use. We shall examine a few *theorems*, that is matters worthy of study, concerning rectangles.

Theorem 1.

The rectangle under one line and the sum of two more lines is the sum of the rectangles under it and of those lines.

Thus if there be a line A B, and sever

M N, O P, the rectangle under A B and the sum of those, which rectangle is written A B.(K L+M N+O P) is the sum of the separate rectangles AB·KL, AB.MN and AB·OP.



In order to show the truth of this theorem we may draw a straight line and measure along it the distances C D, D E, E F equal respectively to K L, M N, O P; then C F is the sum of these. At right angles to C F we make F G equal to A B, complete the rectangle and draw across it perpendiculars at D and E. It is then seen that the whole rectangle C F.F G or (K L+M N+O P).A B is made up of the three rectangles A B.C D, A B.D E, A B.E F.

This corresponds to a very familiar arithmetical proposition, which is exemplified in such a statement as this that seven times three hundred and sixty-five is made up of seven times three hundred, seven times sixty and seven times five or as we may write it

$$365 \times 7 = 300 \times 7 + 60 \times 7 + 5 \times 7.$$

To state this proportion generally we make use of indefinite marks for numbers, thus we may put $a, b, c, d,$ for any four numbers whatever, not for any particular numbers, and may write

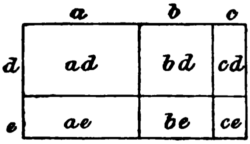
$$(a + b + c)\, d = a\,d + b\,d + c\,d$$

that is d times the sum of a, b and c, is made up of d times a, d times b and d times c.

THEOREM 2.

The rectangle under the sum of one set of lines and the sum of another set is made up of the rectangles under each separate line of the one set and every separate line of the other set.

For the sake of shortness we shall name the lines by single letters. Let then the three lines a, b, c form one set and the two lines d and e another set; the rectangle under $(a + b + c)$ the sum of the first set and $(d + e)$ the sum of the second set is made up of the six rectangles ad, bd, cd, ae, be, ce as is evident from the figure, which needs no explanation.



This agrees with the ordinary process of multiplying numbers; thus the product of 365 by 47 is made up of six separate products, namely 300×40, 60×40, 5×40, 300×7, 60×7 and 5×7.

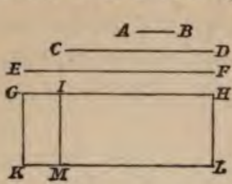
In performing ordinary multiplication the learner is very apt to overlook the reality of what he is doing and to mistake the mere symbol for it: thus he says four times six make twenty-four, not observing that it is in reality forty times sixty with the product twenty-four hundred.

THEOREM 3.

The rectangle under one line and the difference of two other lines is the difference between its separate rectangles with those lines.

Thus, A B being the one line and C D, E F the two others, let us make G H equal to C D the longer of these and *from it cut off* H I equal to the shorter; then G

the difference between the two. Make now G K perpendicular to G H and equal to A B, and complete the rectangle G K L H, also draw I M parallel to G K.



G I K M is the rectangle under A B and the difference between C D and E F; G H L K is the rectangle under A B and E F, while H I M L is that under A B and C D; now G I M K is the excess of G H L M over H I M L, wherefore

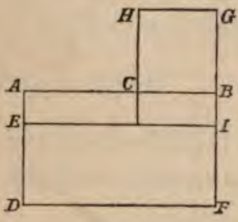
$$AB.(EF - CD) = AB.EF - AB.CD.$$

This theorem is only Theorem 1 stated in a different way.

Theorem 4.

The rectangle under the difference of one pair of lines and the difference of another pair is the sum of the rectangles under the two majors and under the two minors, less the rectangles under each major and the alternate minor.

Thus, in the adjoining figure, A C is the difference between A B and B C, while A E, at right angles to it, is the difference between A D and D E, so that A C K E is the rectangle under the two differences, which is written $(AB - BC).(AD - DE)$. A B F D is the rectangle under the two major lines. Continue F B till B G be equal to D E, and complete the rectangle B G H C, this is the rectangle under the two minor lines B C and D E; and *the whole figure* A C H G F D is the sum of the rectangles



under the two majors and under the two minors. Continue now E K to meet F B in I; then E I F D is the rectangle under the major A B and the alternate minor D E; but I G is equal to A D, therefore K H G I is the rectangle under the major A D and its alternate minor B C; and when these two are subtracted from the whole figure, the rectangle A C K E remains; so that

$$(AB - BC).(AD - DE)$$
$$= AB.AD + BC.DE - AB.DE - AD.BC.$$

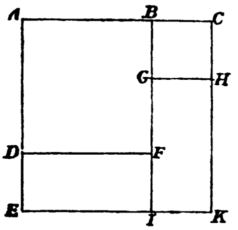
This corresponds to the arithmetical proposition

$$(a - b)\,(c - d) = ac - ad - bc + bd.$$

Theorem 5.

The square on the sum of two lines is made up of the squares of the lines and twice their rectangle.

Since a square is a rectangle with equal sides, this theorem is merely a variety of the second, but, on account of its frequent occurrence, it deserves a special notice.



A B, B C being two straight lines placed so that A C is their sum, let the three squares A B F D, B C H G and A C K E be constructed all on one side of the line; the square of the sum is seen to exceed the sum of the squares by the figure D F G H K E. Let B F be prolonged to meet E K in I, and this figure is divided into two rectangles D F I E, I G H K, each the rectangle under A B and B C.

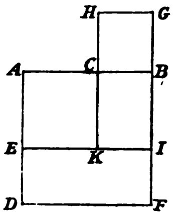
The square of A B, or the rectangle under AB and AB,

may be written A B . A B; it is more frequently written with the sign of *repetition*, viz. a small figure placed above and to the right hand thus A B², as is usual in arithmetical and algebraic writings. Using this form, the above theorem is expressed thus

$$(AB + BC)^2 = AB^2 + BC^2 + 2.AB.BC.$$

Theorem 6.

The sum of the squares of two straight lines exceeds twice their rectangle by the square of their difference.



A B and B C being two straight lines so placed that A C is their difference, let the squares A B F D, B C H G be constructed on opposite sides of the line so that A D F G H C is their sum; make also A C K E the square of A C within the larger square, then it is seen that the square of the difference is less than the sum of the squares by the figure E D F G H K E; and on continuing E K to meet F G in I, this figure is divided into two rectangles D E I F and K I G H, each under A B and B C, so that

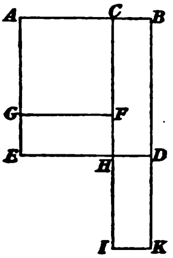
$$(AB - BC)^2 = AB^2 + BC^2 - 2.AB.BC.$$

This is only a case of Theorem 3; the lines A D, D E happen to be equal respectively to A B, B C. The student may observe that the very same letters have been used for the corresponding points of the figures for Theorems 3, 4 and 5.

THEOREM 7.

The difference between the squares of two straight lines is equivalent to the rectangle under the sum and the difference of the lines.

From A B D E the square of A B the longer of two lines, let A C F G the square of A C the shorter be cut out, and there is left, for the difference of the squares, the figure G F C B D E. By producing C F to H we cut this figure into two rectangles, the lesser of which G F H E may be removed and placed in the position I H D K, so as to make up, along with H C B D, the long rectangle I C B K.



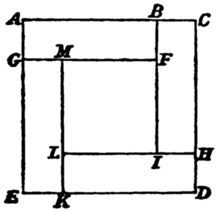
Now B K is the sum of the two lines A B, A C, and C B is their difference, wherefore

$$AB^2 - AC^2 = (AB + AC).(AB - AC).$$

THEOREM 8.

The square of the sum exceeds the square of the difference of two straight lines, by four times their rectangle.

On A C, which is the sum of A B and B C, let the square A C D E be constructed, and in the corners of this let the four rectangles A B F G, C H I B, D K L H, E G M K be placed, and there is formed in the middle M F I L which is the square of M F the difference between A B and B C, wherefore



$$(AB + BC)^2 - (AB - BC)^2 = 4.AB.BC.$$

These four theorems agree with the algebraic formulæ

Theorem	5,	$(a+b)^2=a^2+2ab+b^2$
"	6,	$(a-b)^2=a^2-2ab+b^2$
"	7,	$(a+b)(a-b)=a^2-b^2$
"	8,	$(a+b)^2-(a-b)^2=4ab$

Exercise 1.

The line A B being seven parts, and C F being composed of three parts C D=5, DE=3, E F=1; of what parts is the rectangle under A B and C F made up?

Exercise 2.

C H=9, and I L being the difference between I K=13 and K L = 6, of what rectangles is the rectangle under C H and I L the difference?

Exercise 3.

M P being made up of three parts M N = 15, N O = 11, O P = 6, and Q U being the sum of four parts Q R = 13, R S = 8, S T = 9, T U = 7, of what rectangles is the rectangle under M P and Q U composed?

Exercise 4.

V X being the difference between V W = 17, W X = 8, while Y A is the sum of Y Z = 5, Z A = 4; how is the rectangle V X . Y A to be obtained?

Exercise 5.

B D being the difference between B C = 23 and C D = 15; E G being the difference between E F = 29 and *F G = 18; how* is the rectangle B D . E G to be obtained?

EXERCISE 6.

H K being the sum of H I = 17, and I K = 8, of what parts is its square composed?

EXERCISE 7.

L N being the difference between L M = 19 and M N = 6, how is LN^2 to be got?

EXERCISE 8.

O P being 29, Q R being 27, compute $OP^2 - QR^2$.

EXERCISE 9.

Two lines are 139 and 147 inches respectively, compute the difference between their squares.

EXERCISE 10.

What must be added to the square of 437 to make the square of 438?

LESSON XLVII.

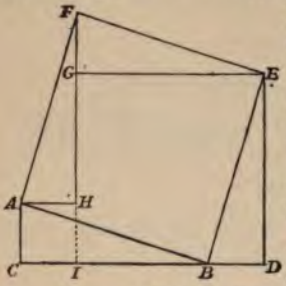
ON RIGHT-ANGLED TRIGONS.

WHEN one angle of a trigon is right, the sum of the two others must be a right angle, and each of them is necessarily acute. The side opposite the right angle is called the *hypotenuse* from the Greek *ὑποτείνουσα* (out-stretcher). The right-angled trigon is the half of a rectangle; it occurs very frequently in business.

The square of the hypotenuse of a right angle is *equivalent to the* sum of the squares of the two sides.

Thus if two lines A C, C B be drawn at right angles, the square of A B which joins their extremities is as much as the two squares of the sides A C and C B; or, as we write it shortly $AB^2 = AC^2 + CB^2$.

Let C B be continued till B D be equal to A C; raise at D a perpendicular D E equal to C B and join B E; the



trigon B D E is clearly a copy of A C B, and B E is equal to A B. But the two angles C B A and C A B, or the two C B A and E B D make together one right angle, wherefore B E is perpendicular to A B, since the three angles at B make together half a turn. Draw E F parallel to B A, A F parallel to B E; A B E F is the square of A B. Draw now F I parallel to E D, A H and E G parallel to C D; the trigons A H F and F G E thus formed are copies of A C B.

If from the square A B E F we cut off the two trigons F G E, A H F and instead of them put A C B, B D E, we get the figure C A H G E D which is made up of C A H I the square of A C and I G E D the square of C B; and thus the square of A B contains as much surface as the squares of A C and of C B put together.

This is the most useful theorem in the whole science of geometry, because it serves to connect that science with arithmetic.

When the two sides are given in numbers we are enabled to compute the hypotenuse. Thus if A C be five parts, its square contains 25 square units; if B C be twelve, its square contains 144 square units, wherefore the *square of A B must* contain 169. Hence if we can discover

that number which when multiplied by itself gives 169, we shall know the length of A B. On trial we find this number to be 13, and conclude that A B is 13 linear units.

In order to make use of this geometrical theorem we must learn how to discover that number which, when multiplied by itself, shall produce a given number; or as it is said, to extract the square root; in the above example we had to find the square root of 169.

In all cases the square root of a number is found by trial, generally by a succession of trials, and it becomes useful to contrive means for lessening the labour thereof. One of the most convenient and most obvious expedients for helping us in this, is the construction of tables of *square* numbers, that is of second powers. Extensive tables of this kind have been computed and published, but the student who desires thoroughly to master the subject should prepare a table for himself before having recourse to printed works.

In one column we place the roots arranged in order; in an adjoining column we place their squares. When the numbers are small the labour is light, but as we come to large numbers the multiplication of each one by itself grows to be tiresome. We may greatly lessen the labour by attending to the manner of the growth of the successive squares.

In order to get the square of a number, say *seven*, we make a row of *seven* dots, repeat that row seven times, and count how many dots there are in the seven rows; there are forty-nine.

To get the square of the next number *eight* we should proceed in the same way, only we have already these *forty-nine dots* made; they may be useful to us; indeed

we have now to place one dot at the end of each row (seven new dots), and then to make a new row of eight dots (in all fifteen new marks), to make up eight times eight. Hence by adding 15 to 49 we get 64 the square of 8. In the same way by adding 8 + 9, or 17, to 64 we obtain 81 the square of 9. And thus we may make a table of squares by addition, as is seen in this example

$$\begin{array}{rcr} 7^2 & = & 49 \\ & & 15 \\ \hline 8^2 & = & 64 \\ & & 17 \\ \hline 9^2 & = & 81 \\ & & 19 \\ \hline 10^2 & = & 100 \end{array}$$

But a much neater arrangement is to write the increments in a column at the side. For example, beginning at 40 we have $40^2=1600$; to this we add $40+41=81$, and we write this 81 at the side; after a very little practice we learn to add the numbers thus placed. The arrangement is shown below, viz.

Root.	Square.	Diff.
40	1600	81
41	1681	83
42	1764	85
43	1849	87
44	1936	89
etc.	etc.	etc.

When this work has been carried on to the root 50 we verify the result, which should be 2500; so that this mode of forming the table while much less laborious than the simpler process of squaring each separate number, gives results much more to be depended on.

This leads me to remark that a common saying "*simple processes are the best*" is not true in this case, and

that it is almost always untrue. We introduce complications or new principles for the purpose of making the process easy or certain. Here the complication of *increments* is brought in, to the great improvement of the operation.

The student should make his own table of squares up to several hundred; or if he have printed tables he may extend these farther for his own use.

The differences of the successive square numbers form the series of *odd* numbers 1, 3, 5, 7 and so on.

When the number whose square root is wanted is found in the column of squares, the root is at once known; if it be not there found exactly, the root cannot be an integer number, it must lie between the roots of the two squares one of which is more and the other less than the proposed number. If it be beyond the limits of the table, its root must be found in some other way; and if the table be not at hand we must have recourse to the method of trial.

For the sake of an example let us suppose the two sides of a right angle to be 36 312 and 33 695 inches, and let us proceed to compute the length of the hypotenuse.

The squares of these numbers are 1 318 561 344 and 1 135 353 025 respectively; and the sum 2 453 914 369 expresses the number of square inches in the square of the hypotenuse. We have therefore to extract the square root of this last number, which has been taken large in order that all the steps of the process may be clearly seen.

When any number is multiplied by 10, the square is augmented 100 times; thus the square of 4 being 16, the square of 40 is 1600, the square of 400 is 160 000, and so on; that is to say, for every step by which the root is removed from the unit's place, the square is *removed* two steps. Now the above numbe

more than 1 600 000 000, therefore its root must be more than 40 000; the same number is less than 2 500 000 000, wherefore its root is less than 50 000. Thus the root of which we are in search is between 40 000 and 50 000, much nearer to the latter than to the former.

In this way we have made sure that the first digit of the required root is 4.

In order to discover the second digit, we examine by how much we are wrong: on subtracting the square of 40 000 from the given number, we find the error 853 914 369; and our business now is to find what must be added to 40 000 to cause its square to increase by this much.

$$\begin{array}{r} 2\,453\,914\,369 \\ 1\,600\,000\,000 \\ \hline 853\,914\,369 \end{array}$$

Let us make a trial and suppose 7 to be the second digit; we shall then have to compute by how much the square of 47 000 exceeds the square of 40 000. Now it has been shown in Theorem 7 that the difference between two squares is the rectangle under the sum and the difference of the two sides; wherefore writing, as in the margin, the numbers 40 000 and 47 000 we take their sum and their difference; the product of these, viz. 609 000 000 is not so much as our error, and therefore the required root is more than 47 000. Let us then try 8 for the second digit; following the same process we have 88 000 for the sum and 8000 for the difference, the product of which, viz. 704 000 000 is still too little. On trying 9 for the second digit, that is on taking 49 000 for the root, we find the sum and the difference as in the margin, giving the product 801 000 000, which still is too small; and thus we are sure that the required root is more *than 49 000*, while it is less than 50 000; the second

$$\begin{array}{rl} 40\,000 & \\ 47\,000 & \\ \hline 87\,000 & \text{sum} \\ 7\,000 & \text{diff.} \end{array}$$

$$\begin{array}{rl} 40\,000 & \\ 49\,000 & \\ \hline 89\,000 & \text{sum} \\ 9\,000 & \text{diff.} \end{array}$$

digit must be 9. As far as we have gone the calculation may be concisely written thus

40 000	2 453 914 369	
40 000	1 600 000 000	first error
89 000	853 914 369	
9 000	801 000 000	
	52 914 369	second error

where by subtracting 801 000 000 from the first error we find 52 914 369 for the remaining error. The discovery of the third digit is made exactly in the same way. Let us try 4, that is to say, let us try the root 49 400. We have to compute the excess of the square of this number above the previous square of 49 000; following the same process as before we find the sum to be 98 400, while the difference is 400; the product of these, viz. 39 360 000 is too small; we may try in the same way 5 for the third digit; this also we find to be too small, but on trying 6, as in the margin, the difference of the squares is found to be 59 160 000 which is too much; hence we know that the required root is between 49 500 and 49 600; its third digit then is certainly 5. The calculation then stands

49 000	
49 400	
98 400	sum
400	diff.

49 000	
49 600	
98 600	sum
600	diff.

40 000	2 453 914 369	
40 000	1 600 000 000	
89 000	853 914 369	first error
9 000	801 000 000	
98 500	52 914 369	second error
500	49 250 000	
	3 664 369	third error

showing the error to be 3 664 369. By continuing this *process we reduce the* error at each step, and ultimately

find 41 537 for the root, without any error, as is shown below, where, for clearness, the whole calculation is re-copied.

40 000	2 453 914 369
40 000	1 600 000 000
89 000	853 914 369
9 000	801 000 000
98 500	52 914 369
500	49 250 000
99 030	3 664 369
30	2 970 900
99 067	693 469
7	693 469

In the actual calculation we save room and obtain greater convenience by omitting the zeroes of position; the above operation is more compactly put

4	2 453 914 369	49 537
4	1 6	
89	853	
9	801	
98 5	52 91	
5	49 25	
99 03	3 664 3	
3	2 970 9	
99 067	693 469	
7	693 469	

where only those digits of the errors which are needed are brought down.

The extraction of the square root occurs so very often that it is worth while to acquire expertness therein. We *may shorten the* work by help of a table of squares; thus

if we possess such a table up to 1000, we may save labour at the beginning by seeking among the squares for the next below 245 391, this is the square of 495, and we have now three digits of the root; the extraction would then take this form

49 5	2 453 914 369	49 537
49 5	2 450 25	
99 03	3 664 3	
3	2 970 9	
99 067	693 469	
7	693 469	

In this way we have found that if the two sides of a right angle be 36 312 and 33 695 inches, the hypotenuse must be 49 537 exactly.

It seldom happens that the root comes out exactly; most commonly there is a remaining error, which has to be reduced by adding fractional parts to the root. This is done by continuing the calculation below the unit's place on the arithmetical scale, that is to tenths, hundredths, thousandth parts of the unit.

When the hypotenuse and one side are given we compute the length of the other side by subtracting the square of the known side from the square of the hypotenuse. The remainder is the square of the side sought; its square root has to be calculated. In this case, however, the labour of the squaring may be avoided; we may multiply the sum of the two numbers by their difference, and so obtain the difference of their squares.

The convenience of this process may be best seen from an example in large numbers. Thus let the hypotenuse be 866 713 inches, and one side 796 775. Instead of squaring these and taking the difference of their squares, we prefer

to take the sum and the difference of the given numbers and to multiply these, thus

866 713	
796 775	
1 663 488	sum
69 938	diff.
13 307 904	
49 904 64	
1 497 139 2	
14 971 392	
99 809 28	
116 341 023 744	diff. of squares.

3	116 341 023 744	341 088
3	9	
64	26 3	
4	25 6	
681	741	
1	681	
682 08	60 023 7	
8	54 566 4	
682 168	5 457 344	
8	5 457 344	

giving 341 088 inches for the other side.

This artifice of converting the difference between two squares into a rectangle or product will be frequently used in the course of our studies.

Exercise 1.

Having constructed a right-angled triangle, make squares on its three sides all outwardly. Find the middle of the mean square, and through that point draw two lines parallel *to the sides of* the square on the hypotenuse, thus making

four equal portions. These portions and the least square may be placed so as to make up the square of the hypotenuse. *This very neat arrangement was proposed by Arthur Perigal, Esq., of London.*

EXERCISE 2.

The two sides of a right angle being 3 and 4 inches, compute the length of the hypotenuse.

EXERCISE 3.

When the sides of a right angle are 5 inches and 12 inches, what is the subtense?

EXERCISE 4.

A board is 4 feet broad by 4 feet 7 inches long; required the distance from corner to corner.

EXERCISE 5.

A room is 13 feet long by 11 feet 1 inch broad; required the diagonal.

EXERCISE 6.

A plot of building ground is 593 feet 4 inches by 526 feet 9 inches, what is the distance cornerwise, if the plot be quite rectangular?

EXERCISE 7.

The hypotenuse of a right angle being 5741, and one of the sides 4060 inches, required the length of the other side.

EXERCISE 8.

Connect three small rings by three strings of the lengths (counting from centre to centre) 9 feet 11 inches, 10 feet, and 14 feet 1 inch; and show that these when stretched out form a right-angled trigon. This makes a very convenient pocket set-square for marking right angles on the ground.

Exercise 9.

At the middle of a line 1220 feet long, a perpendicular is raised 10 feet 1 inch long; by how much are the two sides of the isoskeles trigon so made, longer than its base?

Exercise 10.

The hypotenuse being 3881, and one side 3081, required the length of the third side.

Exercise 11.

The base being 4140, and the hypotenuse 4141, required the altitude.

Exercise 12.

The two sides being 7740 and 5371, required the length of the hypotenuse.

Exercise 13.

A table is 4 feet $7\frac{1}{4}$ inches long by 2 feet $0\frac{3}{16}$ inch broad, what is the diagonal?

Exercise 14.

The side of a square is $40 \cdot 8$ inches, what is its diagonal?

Exercise 15.

The side of a square being 5 feet 1 inch and $\frac{9}{16}$, required the length of the diagonal.

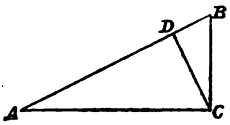
Exercise 16.

The side of an equilateral trigon being $20 \cdot 9$ inches, required its altitude and its area.

LESSON XLVIII.

A RIGHT-ANGLED trigon is divided into two trigons similar to the whole and to each other, by the perpendicular let fall from the vertex of the right angle to the hypotenuse.

A C B being right angled at C, if we draw C D perpendicular to the hypotenuse A B, each of the triangles A D C, C D B is similar to A C B. On comparing A D C with A C B we find the angle at A to belong to both, while the right angle A D C is equal to A C B; and so also of the trigon C D B. Hence the sides of these triangles are proportional and



$$AB : AC :: AC : AD :: CB : CD$$
$$AC : CB :: AD : DC :: CD : DB$$
$$BA : BC :: BC : BD :: CA : CD$$

and thus $AB \,.\, AD = AC^2$, $BA \,.\, BD = BC^2$ and $AD \,.\, DB = DC^2$; because the rectangle under the extreme terms is equivalent to that under the mean terms of a proportion.

Exercise 1.

The sides of right angle being 20 and 15, required the hypotenuse, the perpendicular let fall upon it, and its segments.

Exercise 2.

The sides being 255 and 136, required the perpendicu- let fall upon the hypotenuse, and the parts of that lis

Exercise 3.

The sides being 105 and 88, required the same parts as above expressed in common fractions, and also in decimal parts of the unit.

Exercise 4.

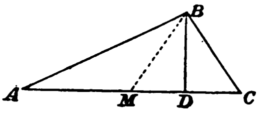
Let the sides be 127 and 43, and compute the hypotenuse, the segments made by the perpendicular, and the perpendicular.

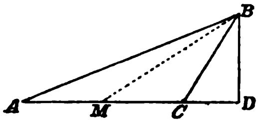
LESSON XLIX.

Part 1.

The difference between the squares of two sides of a trigon is equivalent to the difference between the squares of the distances of the perpendicular from the ends of the third side.

If from the corner B, the perpendicular B D be let fall upon the opposite side, the difference between the squares of A B and of B C is equivalent to the difference between the squares of A D and of D C.





If two unequal magnitudes be both increased or both diminished by the same quantity, the difference remains unchanged. Thus the difference between 9 and 5, is the same as the difference between 49 and 45. Now if the square of *DB be added to* each of the squares of A D and of C D,

the sums are the squares of A B and of C B respectively, wherefore

$$AB^2 - BC^2 = AD^2 - DC^2.$$

If the two sides A B and B C were alike, the perpendicular would fall at the middle of A C; when A B is greater than B C, the perpendicular falls beyond A M, M being the middle point. Now the difference between A D and D C, when the perpendicular falls within as in the upper figure, is the double of M D, because A D is more than A M, or than M C by M D, and M C is more than D C by the same M D, wherefore A D is longer than D C by twice M D. But A C is the sum of A D and D C, wherefore the difference between the squares of A D and of D C is equivalent to twice the rectangle under A C and M D. In the lower figure again, the sum of A D and D C is the double of M D, for if a thread were passed from A to D and back to C, its length would be the sum of A D and D C, but if now the end A be brought to C the thread will be doubled over the whole of M D. In this case A C is the difference between A D and D C, wherefore the difference of the squares of A D and D C is equivalent to twice the rectangle under A C and M D. And thus, whether the perpendicular fall within or without,

$$AB^2 - BC^2 = 2.AC.MD,$$

the difference between the squares of two sides of a triangle is equivalent to twice the rectangle under the opposite side and the distance of its middle point from the perpendicular.

This theorem enables us to compute the distance M D when the three sides of the trigon are known.

For example if the dimensions of the triangle be A B =

884, B C = 741, C A = 845; the difference between the squares of A B and B C is 232 375, which must also be the rectangle under the double of A C and M D; wherefore dividing this number by 1690 we obtain the value of M D which comes out $137\frac{1}{2}$.

$$\begin{array}{r} AB^2 = 781\,456 \\ BC^2 = 549\,081 \\ \hline 232\,375 \end{array}$$

Here A M = $422\frac{1}{2}$, wherefore A D = 560, D C = 285, from either of which we may compute the length of B D, viz. 684; and thus we obtain the area of the figure $\frac{1}{2} \times 684 \times 845 = 288\,990$ square units; here we have the means of calculating the area of a trigon directly from the measurements of the three sides, without drawing it on paper.

The same result should be got *exactly* by drawing a perpendicular from C upon A B, or from A upon B C; whereas the student has already found that the three results obtained by the measurement of the altitudes hardly ever quite agree. Herein we see the advantage of calculation over paper measurement.

Exercise 1.

The sides A B, B C being 15 and 13, with the base A C = 14, how far does the perpendicular fall aside from the middle point? Compute the perpendicular and the area.

Exercise 2.

In the same triangle let fall a perpendicular upon the side 15, and make the corresponding computation, using common fractions.

Exercise 3.

Do the same for the perpendicular let fall upon the side 13.

Exercise 4.

Compute the positions of the three perpendiculars, their lengths, and the area of the trigon having its sides 51, 52, 53, using common fractions.

Exercise 5.

In the case $AB=185$, $BC=95$, $CA=100$, compute the distance MD, the length of DB and the area.

Exercise 6.

Make the corresponding calculations for the perpendicular on BC, and also for the perpendicular on AB.

Exercise 7.

Compute the three altitudes of the trigon having its sides 629, 672, 757.

Exercise 8.

Compute the three altitudes of the trigon whose sides are 43, 740, 765.

Exercise 9.

Make the similar computations for the sides 680, 652, 423.

LESSON XLIX.

Part 2.

The square of the subtense of an obtuse angle is greater than the sum of the squares of the containing sides; the square of the subtense of an acute angle is less than the sum of the squares of the containing sides; in either case by twice the *rectangle* under one of the containing s

and the distance of the altitude upon it from the vertex of the angle.

When A C B is obtuse the altitude on A C falls at D outside of the angle, and A D is the sum of A C and C D, so that $AD^2 = AC^2 + CD^2 + 2.AC.CD$. If to each side of this equality (or equation) we add the square of B D, there results

$$AB^2 = AC^2 + CB^2 + 2.AC.CD,$$

which agrees with the first of the above propositions.

When A C B is acute A D is the difference between A C and C D, and therefore $AD^2 = AC^2 + CD^2 - 2.AC.CD$.

Increasing each side by DB^2 as before, we have

$$AB^2 = AC^2 + CB^2 - 2.AC.CD,$$

which agrees with the second proposition.

These theorems enable us to compute C D directly without reference to the middle point M.

Thus if we have A B = 959, B C = 634, C A = 585, we should compute the squares of the three sides; that of A B exceeds the sum of the others, wherefore the angle subtended by A B must be obtuse.

$AB^2 =$	919 681
$BC^2 =$	401 956
$CA^2 =$	342 225
$AB^2 - BC^2 - CA^2 =$	175 500

The excess of AB^2 above BC^2 and CA^2, viz. 175 500 is twice the rectangle under A C and C D, wherefore dividing 175 500 by 1170 (the double of A C) we obtain 150 for the length of C D.

Therefrom we may compute D B, 616 and thence the area of the triangle, viz. 180 180 square units.

The computation, when the angle at C is acute, is conducted in the same way, only we have $BC^2 + CA^2 - AB^2$.

EXERCISE 10.

Compute the three altitudes and the area of the triangle whose sides are 17, 25, 26.

EXERCISE 11.

The three sides being 24, 34, 53, compute the positions and lengths of the three altitudes.

EXERCISE 12.

The sides being 723, 724, 725, required the altitude on 724.

EXERCISE 13.

Required the positions and lengths of the altitudes of the triangle 91, 250, 289.

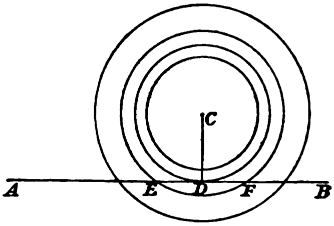
EXERCISE 14.

Also for the triangle 21, 340, 353.

LESSON L.

ON THE TANGENT TO A CIRCLE.

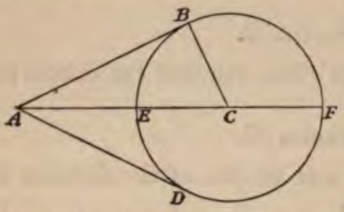
IF from any point C we draw a perpendicular to the straight line A B, that perpendicular C D is the shortest line that can be drawn from C to A B. A circle described from the centre C with a radius shorter than C D cannot reach to A B. If the radius be longer than C D, the circumference of the circle crosses A B twice, once on each



side of D, as at E and F, so that if the line C D and the

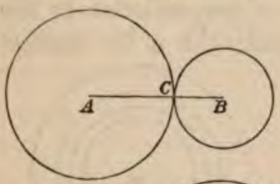
length of the radius be known, we can compute the distance E F, since C D E is a right-angled trigon. But if C D itself be taken as the radius, the circumference will touch the line A B only at the point D. This circle and the line A B are said to be tangent to each other.

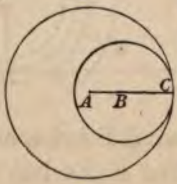
From any point A, taken without a circle, two tangents A B, A D may be drawn to it; the actual operation of



drawing these being like that of joining two points, with this distinction that it is not easy to judge of the exact termination of the tangent, because the two lines run gradually into each other. The exact point B may be found by drawing a perpendicular from the centre.

Since A B C is a right angle, $AB^2 = AC^2 - CB^2$, wherefore if the radius and the distance A C be known the length of A B may be computed. Instead, however, of taking the squares of A C and of C B, and then the difference of those squares, we may compute the rectangle under A F (the sum of A C and C B), and A E the difference of the same lines, putting $AB^2 = AE.AF$.





When two circles meet in the straight line joining their centres, they touch each other externally, that is each one lies entirely without the other; their boundaries only touch at one point. But when, as in the second figure, the circumferences meet at

C in the continuation of the straight line joining the centres A and B, the contact is internal, the circumferences having in common only the single point C. Hence the point of contact of two circles is definitely found by drawign a straight line through their centres.

EXERCISE 1.

The perpendicular C D being 84, compute the half chords (D E or D F) of circles described with the radii 85, 91, 105, 116, 140, 159, 205, 259, 300, 445, 591, and 884.

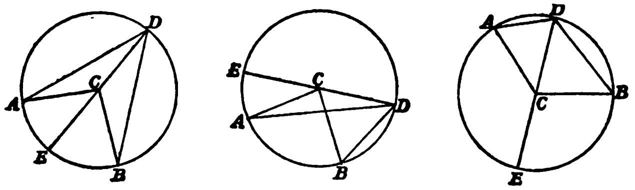
EXERCISE 2.

The radius of a circle (C E) being 72, compute the lengths of tangents (A B) drawn from A when E A is 3, 6, 18, 25, 48, 81, 98, 150, 256, 363, and 578.

LESSON LI.

AN arc of a circle subtends an angle having its vertex at the centre, double of an angle having its vertex in the circumference.

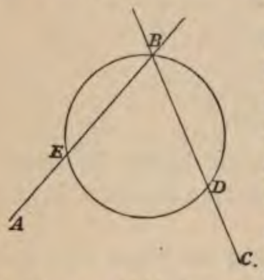
If the extremities of the arc B A be joined with the centre C, and also with any point D in the remaining



part of the circumference, the angle B C A is double of *BDA*.

For if we join D C and produce it to E, the line C E may fall within the angle B C A as in the first figure or without it as in the second; it may also lie along one of the sides of the angle. In any case the angle B C E being equal to the sum of the interior angles B D C and C B D is double of B D C while, for a like reason, E C A is double of C D A; wherefore B C A which is the sum or the difference of B C E and E C A, is double of B D A which is the sum or the difference of B D C and C D A.

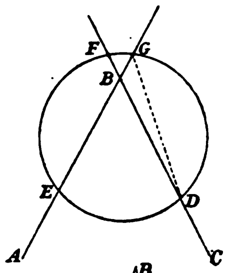
This theorem may be neatly stated thus: "An angle at the circumference of a circle is measured by half the intercepted arc." If we suppose the circumference to be divided into degrees, the number of such degrees in the arc B A gives the value of the angle B C A, wherefore the half of that number indicates the angle B D A. Hence we may use a protractor to measure an angle without placing the centre on the vertex. Thus, in order to measure the angle C B A, we may place the edge of the protractor at the vertex B and count the number of degrees in the arc D E; the half of this number gives the value of the angle C B A.



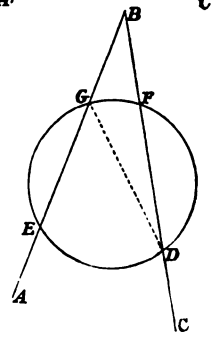
But we do not even need to place the edge of the protractor at B; we may place the instrument so that its edge shall cross each of the lines twice, the vertex of the angle being either within or without the circle.

When, as in the first figure, B is within the circle, the half of the arc D E measures the angle D G E *at the circumference*; while the half of F G measures

F D G; now the sum of the two angles D G E and F D G is C B A, wherefore C B A is measured by *half the sum* of the arcs D E and F G.



When, as in the second figure, the vertex B is without the circle, D B A is the difference between D G E and F D G, so that the angle at B is measured by half the difference of the intercepted arcs D E and F G.



In this way we may measure the angle of two lines whose point of meeting is beyond the paper. When the arcs D E, F G differ little from each other the point of meeting must be at a considerable distance, and when F G is equal to D E the lines D F and E G must be parallel.

EXERCISE.

Draw a number of angles on paper and measure them in various ways by help of the protractor.

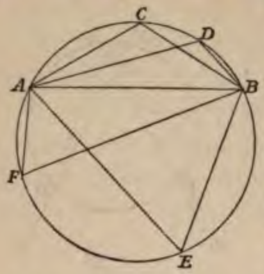
LESSON LII.

ALL angles in the same circular segment are equal to each other.

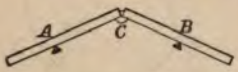
A straight line drawn across a circle divides it into two parts which are called *segments* (*seco* I cut).

When the line passes through the centre the two segments are alike, *each* of them is a semicircle.

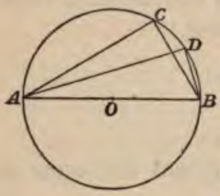
The angles A C B, A D B drawn in the same segment A C D B are equal to each other, because each of them is half of the same angle at the centre, or is measured by half of the same arc B E F A. In the same way A E B is equal to A F B; and it is also to be observed that B C A and A F B being the halves of two angles which make up a whole turn at the centre, must themselves make up half a turn; or as it is usually said, "the opposite angles of a tetragon inscribed in a circle make together two right angles."



This theorem gives us a ready means of drawing flat circular arcs whose radius would be too long for the compass.



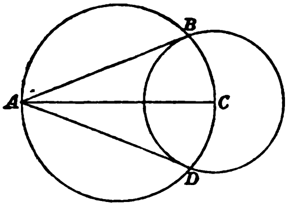
Two rules are jointed at C so that they may be set and secured at the desired angle. The edges of these are brought to bear gently against two pins or two edges set up at A and B. The centre at C is pierced to receive a pencil and the instrument is slid along, the obstacles at A and B remaining fixed. The pencil then traces a circular arc. In this way small parts of very large circles may be traced.



If the line A B be drawn through O the centre of the circle, each of the angles A C B, A D B being half of A O B, is right; or as we say "the angle in a semicircle is a right angle."

Many applications of this theorem occur in the arrangement of instruments and of machines, as well as in geome-

trical processes. A single example may serve to explain the matter. Let it be required to draw from the point A, a tangent to the circle whose centre is C.



On joining A C, and describing a circle with A C, as its diameter, we get B and D the two points of intersection which are clearly the ends of the two tangents.

EXERCISE 1.

On a chord 63 make a circular segment containing an angle of 130°.

N.B. In this case the sum of the two angles C A B and A B C, or D A B and A B D must be 50°.

EXERCISE 2.

On a chord 48 make a circular segment containing an angle of 90°.

EXERCISE 3.

On a chord of 50 make a circular segment containing an angle of 70°.

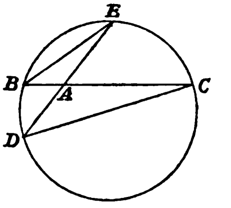
EXERCISE 4.

On a chord of 2 inches make a segment containing an angle of 140°.

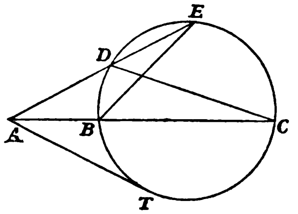
LESSON LIII.

If through any point two straight lines be drawn across a circle, the rectangle under the distances intercepted on the one is equivalent to the rectangle under the distances intercepted on the other line.

If through the point A two straight lines be drawn each cutting the circumference of a circle twice, as B A C, D A E; the rectangle under the distances A B and A C on the one, is equivalent to the rectangle under A D and A E, the distances from A intercepted on the other line.



For if B E and D C be joined, the angle B E D is equal to B C D, each being measured by half of the arc B D; and hence each angle of the trigon B A E is equal to the corresponding angle of D A C; the sides of these trigons are therefore proportional and



$$AB : AD :: AE : AC$$

so that the rectangle under the extremes A B, A C is equivalent to the rectangle under A D, A E the means.

Exercise 1.

The chord of a circular arc is 68 inches, and the breadth of the segment at the middle is 17, required the radius of *the circle.*

EXERCISE 2.

The chord being 68 and the rise at the middle being 4, what is the radius?

EXERCISE 3.

With the same chord and the rise 2, required the radius.

EXERCISE 4.

The rise being only 1 inch with the same chord, compute the radius.

EXERCISE 5.

When the rise is reduced to one tenth part of an inch, what is the radius?

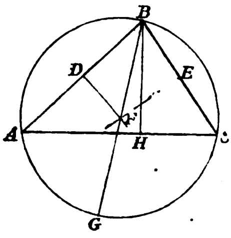
EXERCISE 6.

The span of an arch is to be 36 feet, and the rise in the middle to be 9 feet, required the radius of the circle.

LESSON LIV.

TO DESCRIBE A CIRCLE THROUGH THE THREE CORNERS OF A TRIGON AND TO COMPUTE THE DIAMETER.

THE centre of the circle must be equally distant from the three corners A, B and C; now if we bisect A B at D and there raise a perpendicular, any point in that perpendicular is as far from A as from B; and if, having halved B C at E, we raise at E a perpendicular to B C, any point in that line is as far from B as from C; wherefore the point F at which these two perpendiculars meet must be equally distant from A, B and C; F then is the *centre of the circle* circumscribing the trigon.



Let B F be joined and produced to G, join also A G and let fall B H perpendicular to A C; then since B A G is a right angle, being in a semicircle, and since A G B is equal to H C B, being in the same circular segment, the triangle A B G is similar to H B C and

$$HB : BC :: AB : BG.$$

Now if the three sides be given in numbers, we can compute the altitude B H, so that the diameter B G of the circumscribing circle may be computed.

Hence it follows that the rectangle under A B and B C, two sides of a trigon, is equivalent to the rectangle under B H, the perpendicular let fall on the third side and B G the diameter of the circumscribed circle.

Exercise 1.

Construct a trigon with the sides 51, 38, 25, describe a circle through its three corners and compute the diameter thereof.

Exercise 2.

Describe a circle about the triangle made with the sides 35, 29, 8, and compute the diameter.

Exercise 3.

Compute the diameter of the circle circumscribing the trigon whose sides are 232, 219, 187.

Exercise 4.

The three sides being 769, 600 and 481, to compute the diameter of the circumscribing circle.

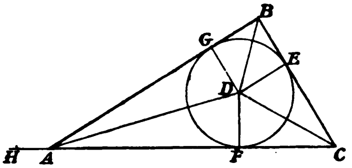
Exercise 5.

The sides being 325, 164 and 163, what is the diameter of the circumscribed circle?

LESSON LV.

TO DESCRIBE A CIRCLE TOUCHING THE THREE SIDES OF A TRIGON AND TO COMPUTE ITS RADIUS.

Two tangents drawn to a circle from a point without it make equal angles with the line drawn from that point to the centre; wherefore if the angle B A C be bisected the centres of circles touching both sides A B and A C must lie in the bisecting line. Again the line bisecting the angle A C B must contain the centres of circles touching both C A and C B; wherefore the



point D at which these two bisecting lines meet, must be the centre of a circle touching all the three sides. On drawing the perpendiculars D E, D F, D G, we form the triangle D E C equal to D F C, and D F A equal to D G A; so that D E, D F, D G are all alike, and a circle described from D with the radius D F touches the three sides at E, F and G.

On joining B D we again form two equal triangles B E D, B G D, wherefore B D bisects the angle A B C; hence this theorem "the lines which bisect the three angles of a trigon meet in one point, which point is the centre of the inscribed circle."

Beginning at the point E and proceeding round the boundary of the triangle we find that boundary or *perimeter* (measure round) made up of six parts E C, C F, F A, A G, G B and B E, or of twice E C, twice F A and twice G B; so that the half boundary or *semiperimeter* is made up of E C, F A and G B. If then we prolong C A till A H be *equal to G B,* the line H C is the semiperimeter of the

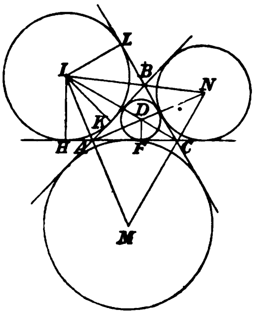
trigon, that is half the sum of the three sides A B, B C, C A. Also A H or G B is the excess of the semiperimeter above A C, E C is the excess of the semiperimeter above AB, and F A is its excess above B C. Hence when the three sides are given in numbers, we may find the points E, F and G by computation.

For example, let $CA = 87$, $AB = 65$, $BC = 44$; the computation may be arranged conveniently as in the margin.

C A = 87	11 = G B
A B = 65	33 = E C
B C = 44	54 = F A
Sum = 196	98 = C H

The sum of the three sides being taken, its half is written in the same line, and above the half sum, its excesses above the sides are written opposite to the respective sides. The sum of these three excesses should make up the semi-perimeter.

Besides the one circle touching the three sides internally, there are three circles placed outside of the trigon and touching its sides or their continuations.



Thus if we produce C D to meet a line bisecting the angle B A H, in I, I is the centre of a circle touching A B, and the continuations of C A and C B. For if we draw the perpendiculars I H, I K, I L, we perceive that I A H is equal to I A K, so that I H is equal to I K and AH to A K; also that I H C is equal to I L C, giving I H equal to I L, H C equal to L C. It is also obvious that the line I B bisects the angle A B L, and that K B is equal to B L.

In the same way we may obtain the centres M

and N of two circles outside of A C and of C B respectively.

If we go round the triangle beginning at K, we find the perimeter to be made up of K B, B C, C A and A K; but instead of K B we may put L B and for A K we may put A H, so that the perimeter is the sum of L B, B C, C A, A H, that is of L C and C H; but these are alike, wherefore C H is the semiperimeter.

Since I A B is the half of H A B, and B A D the half of B A C, it follows that the whole angle I A D is a right angle and that A I H is equal to D A F; wherefore the trigons A I H and D A F are similar, so that I H : H A :: A F : F D, and consequently the rectangle under I H and F D is equivalent to that under H A and A F.

Also the trigon I H C is similar to D F C, whence H C : F C :: I H : D F. But the ratio of I H to D F is the same as that of two rectangles having these lines for their lengths and having a common breadth: make that common breadth D F and we have I H : D F :: I H.D F : D F.D F wherefore H C : F C :: I H.DF : D F.D F; but instead of I H.D F we may write its equivalent H A.A F and obtain the proportion

$$HC : FC :: HA.AF : DF^2.$$

Now when the three sides are given in numbers, the semiperimeter H C and the three excesses H A, A F, F C are easily computed, and thus the square of the inscribing radius D F may be had. Hence the formula

$$DF = \sqrt{\left\{\frac{HA.AF.FC}{HC}\right\}}$$

or the rule "multiply together the numbers representing t *three excesses, divide* the continued product by the s

perimeter and take the square root of the quotient, to get the number expressing the radius of the inscribed circle."

In the preceding example we have

$$DF = \sqrt{\frac{54.33.11}{98}} = \sqrt{\frac{9801}{49}} = \frac{99}{7} = 14\tfrac{1}{7}.$$

EXERCISE 1.

The three sides of a triangle being 17, 10, and 9, compute the radius of the inscribed circle, and also the radii of the three circles of external contact.

EXERCISE 2.

The sides being 56, 39, 25, what are the radii of the four circles each touching the three sides or their continuations, and what the radius of the circumscribing circle?

EXERCISE 3.

Required the radii of circles touching the three sides 185, 109 and 84; as also the radius of the circumscribing circle.

EXERCISE 4.

Compute the radii of the inscribed circle, of the circles of external contact, and of the circumscribed circle of an equilateral trigon on the base 724.

EXERCISE 5.

Required the corresponding radii for the trigon 724, 725, 723.

LESSON LVI.

Porism.

TO COMPUTE THE AREA OF A TRIGON WHEN THE THREE SIDES ARE GIVEN.

A porism is some advantage gained in the course of study ($\pi o \rho \acute{\iota} \sigma \mu \alpha$); in the present instance we get a very neat and rapid calculation of the area of the trigon, from the study of its inscribed circle.

Referring to the first figure in the previous lesson, the triangle A D F is half the rectangle D F . A F, so that the tetragon A G D F is equivalent to D F . A F; in the same way the tetragon C F D E = D F . F C, and B E D G = D F . B E = D F . H A. Now these three tetragons make up the whole trigon A B C, while the three rectangles D F . A F; D F . F C; D F . H A make up the single rectangle D F . H C; so that "the area of a trigon is equivalent to the rectangle under its semiperimeter and the radius of the inscribed circle."

If then we compute the inscribing radius by help of the preceding theorem, it is only needed that we multiply its numerical value by that of the semiperimeter in order to get the expression for the area.

When, however, we desire to know the area alone, we may avoid the actual computation of the inscribing radius, because

$$\sqrt{\left\{\frac{HA.AF.FC}{HC}\right\}} \times HC = \sqrt{\{HA.AF.FC.CH\}}.$$

Or we may reason thus:

$$IH : HC :: DF : FC.$$

Taking the rectangles under the terms of the first ratio and DF; and the rectangles under the terms of the second ratio and HC, this proportion becomes

$$\text{IH.DF} : \text{HC.DF} :: \text{HC.DF} : \text{HC.FC}, \text{ or}$$
$$\text{HA.AF} : \text{ABC} :: \text{ABC} : \text{HC.FC};$$

that is to say, "the surface of a trigon is a mean proportional between the rectangle under two of the excesses and the rectangle under the semiperimeter and the third excess," hence

$$\text{ABC} = \sqrt{\{\text{AH.AF.FC.CH}\}}.$$

In the preceding example therefore the area is given by the arithmetical operation

$$\sqrt{\{11.33.54.98\}} = 1386\,.$$

If we put a, b, c for the three sides of a trigon its area is given by the formula

$$\sqrt{\left\{\frac{a+b-c}{2} \cdot \frac{a-b+c}{2} \cdot \frac{b+c-a}{2} \cdot \frac{a+b+c}{2}\right\}}.$$

Exercise 1.

Compute the area of a triangle having its sides 979, 925 and 714.

Exercise 2.

The student may re-compute, by help of this theorem, the areas of the figures which he has already obtained by other means.

Exercise 3.

Compute the areas of the following trigons:—332, 809, 975; 572, 999, 977; 969, 890, 193; 910, 901, 61; 843, 707, 340; 748, 709, 93; 584, 557, 173; 441, *156*; *231*, *185*, 130.

EXERCISE 4.

Compute the area of the convex hexagon A B C D E F, having A B = 388, B C = 701, C D = 304, D E = 471, E F = 459, F A = 557, A C = 939, C E = 745, and E A = 820. Construct also a square having the same area.

EXERCISE 5.

Compute the area of the decagon A B C D E F G H I J, having A D = 724, D H = 723, H A = 725, A C = 365, C D = 363, D F = 481, F H = 308, H J = 408, J A = 324, D E = 356, E F = 135, F G = 205, G H = 117, H I = 218, I J = 209, A B = 229, B C = 156, and compute the side of a square equivalent thereto.

LESSON LVII.

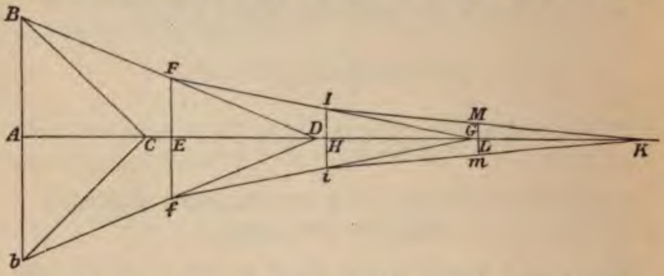
THE CIRCUMFERENCE OF A CIRCLE BEING GIVEN, TO COMPUTE ITS RADIUS.

CIRCLES and round bodies occur so often in business, particularly in mechanical matters, that we must know how to make computations concerning them; and we must be careful to understand how the calculations are made. The following very beautiful and simple process is taken from the third edition of John Leslie's 'Elements of Geometry,' Edinburgh, 1817.

The length of a string being given, we wish to make a circle round which that string may fit exactly. For this we begin with a square or regular four-sided figure; the length of each side must be the quarter of the length of the string. From the square we get a regular octa *having the same* length of boundary, each side

octagon being half of the side of the square. From the eight-sided we get a sixteen-sided regular figure, from that a thirty-two-sided figure and so on, until the sides become so numerous and so small that their aggregate is not distinguishable from the circumference of a circle. In this way we never can reach *absolutely* the size of the circle, but we may easily get near enough to the truth for all business purposes.

Having drawn from A an indefinite straight line we place at right angles to it A B, A *b* each equal to the eighth part of the given circumference; then making A C, equal to A B and joining C B, C *b*, the angle B C *b* is right



and would lie four times round the point C; and, if the trigon B C *b* were so repeated, a square would be formed having C for its middle point, C A being the radius of the inscribed, and C B the radius of the circumscribed circle, the one circle being too small, the other too large for our purpose.

Make now C D as long as C B, and join B D, *b* D. The angle *b* D B is half of *b* C B and so may be laid eight times round the point D. If the triangle B D *b* were so repeated a regular octagon would be formed, whose perimeter would be double of the prescribed length, wherefore in order to get an octagon of the proper size we shall halve A D in E,

and draw the perpendicular F E f; then the regular octagon formed by placing F D f eight times around D would have its boundary of the prescribed length, D E being the radius of the inscribed, D F that of the circumscribed circle, the former being too small, the latter too large, and each of them being much nearer to what is wanted than the former pair were.

Making D G equal to D F, joining F G, f G, bisecting E G in H and drawing the perpendicular I H i, we get I G i the sixteenth part of a regular sixteen-sided figure with G for its middle point, G H for its inscribing and G I for its circumscribing radius. These two differ very little from each other, one being shorter, the other longer than the radius for which we are seeking.

Proceeding in the same way we obtain M K m, the thirty-second part of a regular figure of thirty-two sides, whose inscribing and circumscribing radii K L and K M differ by so little that we find it difficult to continue the work farther on paper; the mean between them may then be sufficiently near the truth for paper work.

But although the accuracy of actual delineation be thus soon exhausted, we may carry on our calculations much farther, so far indeed as to bring the error to be less than any small fraction that may be determined on, less than the thousandth part, than the millionth part of the linear unit.

The mode of computation is very simple, the labour of it is trifling. Since A B and A C are both known, B C may be computed; D E is half the sum of C A and C B, and E F is the half of A B; these being found, we compute the circumscribing radius D F, and thus by calculations consisting of squarings and extractions of the square root *we ultimately obtain* the radius of the circle to within the

prescribed degree of precision. By a little arrangement and attention we may much reduce the amount of work.

As an example, we may propose 8800 inches for the circumference of the circle, and one tenth part of an inch as the utmost error allowable in the radius.

In this case $AB=1100$, $AC=1100$, whence $CB^2=2420000$, and $CB=1555 \cdot 6$. Half the sum of CA and CB gives $1327 \cdot 8$ for DE, while EF is 550. On squaring these and taking their sum we find $2065552 \cdot 84$ for the square of DF, whence $DF=1437 \cdot 2$, so that the radius of the required circle is ascertained to be between the limits $1327 \cdot 8$ and $1437 \cdot 2$.

The results of the calculations may be advantageously arranged as in the subjoined scheme.

Num.	Ins. Rad.	Ins. Rad.²	Half side.²	Cir. Rad.²	Cir. Rad.
4	1100·0	1210000·00	1210000·00	2420000·00	1555·6
8	1327·8	1763052·84	302500·00	2065552·84	1437·2
16	1382·5	1911306·25	75625·00	1986931·25	1409·6
32	1396·0	1948816·00	18906·25	1967722·25	1402·8
64	1399·4	1958320·36	4726·56	1963046·92	1401·1
128	1400·2	1960560·04	1181·64	1961741·68	1400·6
256	1400·4	1961120·16	295·41	1961415·57	1400·5
512	1400·5	1961400·25	73·85	1961474·10	1400·5

In the first column we write the number of the sides of the polygon; in the second column we place the inscribing radius; the square of this is written in the third column; the fourth column is for the square of the half side; the sum of these two squares is the square of the circumscribing radius, written in the fifth column; and the root thereof written in the last column is the circumscribing radius itself.

The details of the polygon of four sides as written in the *first line* having been thus found, half the sum of its outer

and inner radii gives the inscribing radius of the eight-sided figure; the details of this figure, found in the same way, are written in the second line; and thus the work proceeds until the outer and inner radii do not differ. It is to be remarked that since E F is the half of A B, EF^2 is the fourth part of AB^2, so that the squares of the half sides are got by quartering.

Thus we have found that when the circumference of a circle is 8800 inches, its diameter is almost exactly 2801 inches; or, omitting the single inch, we may say that the ratio of the circumference to the diameter of a circle is very nearly as 88 : 28 or as 22 : 7.

The student should repeat this calculation very carefully, carrying the work to two or three decimal places farther than in the example and so obtaining a more exact value. He may further vary it by assuming the circumference as 568 000, which should give the radius almost exactly integer.

The thoughtful student will have noticed that we may begin with any known regular polygon. We may take the hexagon, for example, making A B the twelfth part of the proposed circumference and inflecting B C = B *b*. The assumption of 666 000 for the circumference may be convenient in this case, or still better 623 958 with several decimal places.

Whatever number we may assume for the circumference' and with whichever polygon we may begin, the ratio of circumference to the diameter of the circle comes out the same. For ordinary rough purposes we may say that the diameter is to the circumference as 7 to 22. Much more accurately, indeed with precision sufficient for most business purposes, we may take the ratio 113 : 355 which was first *proposed* by Metius. This ratio has been

puted with excessive precision; it is usually and most conveniently expressed by help of decimal fractions. The first of the above-mentioned ratios gives

$$\frac{22}{7} = 3 \cdot 142\,857\,;$$

that resulting from the above calculation is

$$\frac{8800}{2801} = 3 \cdot 141\,735\,.$$

Metius' ratio gives

$$\frac{355}{113} = 3 \cdot 141\,592\,920\,,$$

while the true ratio carried to twelve decimal places is

$$3 \cdot 141\,592\,653\,590\,;$$

so that the ratio 113 : 355 errs by about the four millionth part of the radius.

By using this ratio we may easily compute the diameter from the circumference or the circumference from the diameter. As this number $3 \cdot 1415$ etc. occurs very often, it has become the custom to denote it by the Greek letter π, the first letter of the word *perimeter;* so that if C denote the circumference of a circle and D its diameter we have

$$C = \pi D\,, \text{or } D = \frac{C}{\pi}\,.$$

Exercise 1.

In order to get the diameter of a cylinder, a strip of paper was wrapped round it and the overlapped portion cut through both folds; the length of the strip was then measured and found to be $23 \cdot 69$ inches; what is the diameter?

EXERCISE 2.

The diameter of a circle being 38·73 inches, required its circumference.

EXERCISE 3.

What are the outer and inner circumferences of a ring included between two concentric circles whose radii are 13 and 17 inches?

EXERCISE 4.

The distance between the centres of two circles which touch externally being 22 inches, and the radius of the one being 14 inches, required the radius of the other, the circumference of each, and the sum of the circumferences.

EXERCISE 5.

The distance of the centres being 22 as before, the radius of the larger circle is made 12; what are the results of the change?

EXERCISE 6.

With the same distance between the centres, the touching circles are made alike; what is the sum of the circumferences?

LESSON LVIII.

TO COMPUTE THE SURFACE OF A CIRCLE.

THE area of the triangle m M K is equivalent to the rectangle under L M and L K, wherefore the whole polygon of thirty-two sides of which m K M is a part, is equivalent to the rectangle under its half perimeter and the radius of its inscribed circle. Now the same is true for a figure of any number of sides circumscribing the circle; when the number of the sides is excessively great, the boundary of the polygon becomes almost exactly the

the circle, wherefore we conclude that the area of a circle is equivalent to the rectangle under its half circumference and its radius.

Now the circumference is $3 \cdot 1416$ times the diameter or πD, wherefore, R being put for the radius, πR is the half circumference, and consequently the area (A) is πR^2, that is to say the surface of a circle is somewhat more than three squares of its radius, very nearly $3\frac{1}{7}$.

From the two equations

$$C = 2\pi R, \qquad A = \pi R^2,$$

we get

$$R = \frac{C}{2\pi}, \qquad R = \sqrt{\left(\frac{A}{\pi}\right)},$$

as also

$$A = \frac{C^2}{4\pi}, \qquad C = 2\sqrt{(\pi A)},$$

which serve to find any two of the quantities, A (area), C (circumference), R (radius) when the third is known. If we write for π the very close approximation $\frac{355}{113}$, these six equations may be written

$$C = \frac{710}{113} R, \qquad A = \frac{355}{113} R^2,$$

$$R = \frac{113}{710} C, \qquad R = \sqrt{\left(\frac{113}{355} A\right)},$$

$$A = \frac{113}{1420} C^2, \qquad C = 2\sqrt{\left(\frac{355}{113} A\right)}.$$

Exercise 1.

The radius of a circle being 1 foot, how many square inches are in its area, and what is the side of a square *containing the* same area?

Exercise 2.

A right-angled trigon being constructed with the sides of the right angle 231 and 160; circles are described on its three sides as diameters; compute their circumferences and their areas. The circle on the hypotenuse is equivalent to both the others.

Exercise 3.

Compute the radius of a circle which shall contain exactly 1 square foot of surface.

Exercise 4.

What is the smallest round out of which a triangle with the sides 89, 82, 57 may be cut; and what is the waste?

Exercise 5.

A three-cornered scrap of wood has its sides 8·7; 6·5; 4·4; what is the largest round that can be made out of it, and how much waste is there?

Exercise 6.

Describe a circle which shall have as much surface as an equilateral trigon on the base 52.

END OF PART I.

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